Markus Pantsar, Paula Quinon[1]:

Bootstrapping of the natural number concept: regularity, progression and beat induction

Introduction

Numerical cognition is a field of research that has traditionally been developed mostly by psychologists and neuroscientists. In the past few decades, the growth in both the quality and the quantity of empirical data has been enormous and has enabled researchers to formulate various theories and models of the human processing of numerical information. As one result, the study of numerical cognition is nowadays receiving constantly increasing attention also from philosophers, who see the empirical research as being relevant to such general philosophical questions as what natural numbers are, how we can acquire knowledge about them, and what the relation is between numbers, as used in everyday life, and natural numbers, as defined in formal mathematical theories.

In the philosophy of mathematics, the most influential way of including results from cognitive sciences in the argumentation can be classified as embodiment, belonging mostly to the philosophy of mathematical practice. In this paper, however, we approach our questions from a different philosophical perspective: the *conceptual analysis of language* and *conceptual engineering*. One important endeavor for us is to determine what conceptual content the expressions of a language get from empirical experience, and what comes from interaction with other concepts. In this paper we conduct our analysis in the paradigm presented in the following

[1] The authors are listed in alphabetical order as their intellectual input is equal.

sources (Dehaene 1997/2011; Spelke 2000; Feigenson et al. 2004; Carey 2009).[2] According to this paradigm our developed arithmetical knowledge is rooted in particular cognitive mechanisms, called *core cognitive systems*, which are innate and which we share with many nonhuman animals.

In the field of numerical cognition it is commonplace to believe that a certain subset of innate systems of core cognition form the basis for the number concept to arise. Upon this basis the second source of conceptual content is activated. This source is the conceptual content arising from a web of interrelated concepts, which in turn emerge from historical and socio-cultural processes.

We agree with this general approach and we do not pretend to have a full understanding of the long process from the primitive origins to developed arithmetic. Our objective in this paper is to contribute to the discussion on which conceptual content is necessary for children to construct the first partial meanings that will subsequently lead them to meanings of arbitrarily large natural numbers and the meaning of the concept of natural number in general, concepts widely used in our contemporary Western society. More precisely, we investigate which conceptual content of the concept of natural number can be extracted from the core cognitive systems that the research community today recognizes as being involved in processing quantities. As we evaluate whether this content is sufficient for the number concept to actually develop, we will observe that the exact natural number concept includes characteristics that cannot be derived from the core cognitive systems. In particular, one key feature – the *regularity* of the natural number progression – is usually assumed without providing a cognitive basis for it. In the last section, we propose a solution to this problem based on the idea that there is another cognitive system which is involved in processing discrete items. That system is the sensitivity to *rhythm*.

The research paradigm we are focusing on in this paper is powerful, but it is only one of several currently developing paradigms of number cognition. Moreover, even within this paradigm many unequal hypotheses are proposed to explain the details of the development of the core cognitive content into the exact natural number concept. The first, and probably the most influential, model of this development was set by Stanislas Dehaene (1997/2011). Dehaene's main idea is that quantitative information (how many elements there are in a collection), to which humans are

[2] See Footnote 11 for other paradigms suggested by Gelman & Butterworth (2005) and Leslie et al. (2007).

exposed from a very early age and on daily basis, is processed by an innate core cognitive system enabling the estimation or approximation of these quantities in an analogous manner (a collection of discrete elements correspond to one mental representation; if the cardinality of the collection is different, so is the magnitude of its representation). This cognitive system responds to different modalities of sensory experience: visual, auditory, tactile etc. For instance, when one looks at books on a shelf, the cognitive system responsible for processing quantities forms a representation approximating the quantity of books. Similarly, when one is given an opportunity to choose between two piles of candies, one is able to choose the bigger one without counting the candies. The system functions constantly, both when quantitative information is consciously conceptualised (as a symbolic representation) and when it remains only in the background of, for instance, decision making processes. Dehaene claims that through this constant experience with local estimations and comparisons there emerges a sense of progression of quantities or magnitudes, what he calls the "*number sense*". Thus, number sense is the idea that the system responsible for processing quantities in an approximative manner (called the *Approximate Number System*, or ANS) is also responsible for forming the concept of progression of increasing magnitudes, and that it fully enables creation of the concept of exact natural numbers.

Another view within this paradigm emphasizes the importance of the cognitive mechanism responsible for *subitizing* in the process of number concept acquisition. Subitizing means the ability to determine the quantity of objects in the field of vision without counting. The same mechanism is used for tracking small groups of objects in space and it is limited to values four and less. Unlike the number sense, this P*arallel Individuation System* (PIS) functions by forming individual representations for all apprehended items (Carey 2009; Carey & Sarnecka 2006). For instance, three boxes might be represented by "box", "box", "box", instead of there being a representation for "three". This mechanism is limited by working memory. Children typically first learn the names of the first few numbers by heart in the process of socialisation, forming a so-called "placeholder structure" for numerals. At this stage, however, children do not know how to match numerals with the cardinalities of collections. Instead, the placeholder structure contains symbolic representations which can later be filled in with meaning. When children learn to associate meanings to the elements of the counting list, they do so one at a time and in order. At the age of approximately 3.5 years children acquire a kind of "aha" experience that allows them to generalise over the required knowledge. This experience is related to understanding the so-called *cardinality principle*, which states that adding one element to the set of items means moving one step forward in the list of numerals. This way, children learn the sense of discreteness, as well as an "add one" procedure that is responsible for grasping the successor operation for natural numbers.

In this paper, we will argue against Dehaene, Carey and Sarnecka, as well as other available versions of this paradigm, claiming that their hypotheses of how the number progression is formed are unable to capture one crucial characteristic of the contemporary concept of natural number, which is the *regularity* (or equality of distances) between elements of the number sequence. In addition to corresponding to our intuitions of what natural numbers are, this regularity is also necessary to employ natural numbers as subjects of computation. We will argue that the number sense alone, unlike proposed by Dehaene, does not provide sufficient conceptual content to account for regularity. Neither does the theory of Carey and Sarnecka, whose "add one" procedure does not guarantee that the added elements are all alike. Therefore, the acquisition of the general natural number concept requires further explanation.

But where does the sense of regularity comes from? Is it a basic cognitive capacity, a core system like the Approximate Number System or the Parallel Individuation System? Or is it a consequence of socio-cultural learning? It is widely recognized that the number sense is universal whereas the development of arithmetic is not. There are "non-arithmetical"[3] tribes who never develop numbers words beyond "one", "two" and "many", as shows the example of Pirahã. Yet people in those tribes possess a similar "number sense" to people in arithmetical cultures (Gordon 2004). Similarly, the ability to subitize is considered to be universal (*i.e.* shared by all humans independently of their cultural background, and possibly also shared with animals) and precede arithmetical education (Klein & Starkey 1988).

While presenting our hypothesis that sensitivity to regularity, which very possibly emerges from the sensitivity to rhythm (called *beat induction*), also plays a role in the development of exact natural number concept, we must ask how it relates to the number sense and subitizing. Is sensitivity to rhythm universal? If so, is it used universally in processing quantities? If this is also the case, how is it related to number sense and the ability to subitize? How is it combined with these two? Or perhaps the sensitivity to regularity is neither core nor basic, but rather mediated

[3] By "arithmetical culture", we mean cultures that have developed at least a vague idea of the general concept of natural number. For instance, we will call "arithmetical" a culture, like the Mayans, which uses symbolic representations of numbers in economic exchange or other calculations. We will call "non-arithmetical" a culture, such as the Pirahã or the Munduruku, whose languages do not contain stable names for quantities beyond the first few.

by some cultural input. It can, for instance, emerge in consequence of being exposed to the regular structure of language. Or is it possible that the regular structure of language and the sensitivity to regularity are both manifestations of beat induction?

These are all important questions in pursuing the theory that regularity becomes an essential feature of the structure of natural numbers through our sensitivity to rhythm. To answer them, the hypothesis that seems the most plausible to us, and which we are going to defend in this paper, states that the sensitivity to rhythm is a basic cognitive capacity, an independent conceptual factor. We claim that it is manifest in all systems of tallying, as well as in the contemporary Western *linear progression of natural numbers*. It thus strongly influences our understanding of what natural numbers are, independently of number sense and the ability to subitize.

Whereas number sense is *quasi-logarithmic* in character (*i.e.* it becomes increasingly harder to estimate and distinguish between quantities as they become larger), the key characteristic of the contemporary, Western linear progression of natural numbers (the *Linguistic Number System* or LNS) is that it aims at encoding a *regular* progression of natural numbers. In this paper we suggest that the regularity of LNS is often grasped through its widely-used characterization: a visual *number line*. Representations of natural numbers are *evenly spaced* on a visual number line and this characteristic needs to be grounded in one way or another. Here we address the question as to where it comes from. In the standard number line used in education, the distance from one to two is the same as, say, the one from ten to eleven, or thirty-six to thirty-seven. Neither number sense nor subitizing predicts this. In fact it is common for people from non-arithmetical cultures *not* to place numbers evenly on the line – if they have a conception of number line in the first place (Dehaene 1997/2011; Núñez 2011). That suggests that people from arithmetical cultures use additional conceptual resources in order to account for the meaning of numerals. The hypothesis here is that they can be found in the sensitivity to rhythm and regularity.

This paper is philosophical and our main objective is conceptual, and as such will not provide new empirical data to strengthen the hypothesis. Instead, the purpose is to analyze the concept of natural number as it is understood in arithmetically developed cultures and study how the required conceptual content may emerge. Formulating the hypothesis is only the first step in this work, and we do not want to claim anything further about the sensitivity to rhythm (*e.g.* whether it is a core cognitive system comparable to the number sense of Dehaene).

However, it has been established that this ability manifests itself already in the first days of human life in the sensitivity to rhythm change and syncopation (*beat induction*). As we will see, acuity in the sensitivity to rhythm also manifests itself in other cognitive tasks, such as proficiency in learning language (Dellatolas et al. 2009). It would appear to be exactly the kind of universal ability with established cognitive connections that can provide, at least *prima facie,* the missing factor in explaining the development of the exact natural number concept. As such, the hypothesis is fruitful empirically, as well as – we hope to show – philosophically.

In **Section 1** of this paper we present the paradigm of core numerical cognition, to which we apply conceptual engineering. In **Section 2** we look at the properties of the natural number line, as well as the conceptual content necessary to bootstrap the general concept of natural number. In **Section 3** we discuss natural numbers as a particular case in the context of the general problem of concept acquisition, focusing on Fodor's continuity thesis. In **Section 4**, we argue that the existing theories on the bootstrapping of the natural number concept are incomplete and suggest sensitivity to rhythm and regularity as the further ability needed to ensure the regularity of the emerging number line.

1 Core Cognition and Conceptual Engineering

1.1 Core Numerical Cognition

Empirical studies have firmly established that human sensitivity to quantities develops spontaneously from an early age, long before the onset of verbal counting systems. According to many studies, there is evidence that processing quantities can be observed in various nonhuman animals, as well as in human infants as young as 5 or 6-month old (Wynn 1992; Dehaene 1997/2011; Spelke 2000; Nieder 2011; Hyde 2011). Researchers do not fully agree which cognitive systems are activated first, nor is there any consensus over which systems play the crucial role in triggering the first conceptual understanding. There is even disagreement over the cognitive systems that play any role at all in the symbolic representations of natural numbers. However, one thing that is generally accepted is that there are several cognitive systems for processing quantitative information. In the paradigm that we work in, two preverbal innate (*core*) cognitive systems play a crucial role in the number concept formation. Another part of the conceptual content of numerical representations is triggered by the features of the linguistic system of symbolic representations.

Let us first take a look at the core cognitive systems. Their existence is postulated by developmental psychologists (see Spelke 2000, 2003) to account for the non-symbolic innate abilities that participate in inferential contexts. As such, they are opposed to purely sensory/perceptual representations that are not further processed or computed. Core cognition includes a variety of general and also task-specific cognitive systems for "representing material objects, navigating through the spatial layout, recognizing and interacting with other animals, and the like" (Spelke 2003, 280). These systems serve as building blocks for mature cognitive abilities and for new conceptual content to arise. Developmental psychologists suggest that core systems are at the root of human physics, human mathematics, human politics, law and games (Spelke 2003, 280). Core cognitive systems function throughout the whole individual life span. Similarity with core cognitive systems that are detected in nonhuman animals suggests that these systems have a long and shared evolutionary history, even if animals do not use them to construct similar conceptual structures.

The first core cognitive system for processing quantities, the Parallel Individuation System (PIS), enables humans and other animals to form mental models of small collections, usually up to three or four individuals at the same time. This ability is called subitizing and it enables the simultaneous perception of one to four objects in a parallel manner. Discrete representations are formed by distinct mental representations for each of the objects. The idea that humans process small quantities by way of a subitizing system was presented by Starkey and Cooper (1980). This system is based on the ability to represent objects as persisting individuals (Spelke 2000), and it is closely related to the ability to perform multiple-object tracking (Trick & Pylyshyn 1994). However, rather than being a general system for discrete quantities, subitizing only works for small quantities due to constraints such as the available space in working memory. Thus, while it can be used for determining quantities, PIS as a core system is limited. In addition, it is not quantity-specific as it serves many other purposes, such as tracking objects in space.

As subitizing is a pre-verbal ability, quantity is represented only implicitly in PIS. The PIS model contains no symbol that means, say, three. However, when young children learn the meanings of the number words "one", "two", "three" and "four", they appear to connect these words to the existing PIS-based mental models of one, two, three and four objects. Thus, these number words become children's first symbols for exact cardinalities, and it is plausible that PIS provides reference for them.

The second core system, the Approximate Number System (ANS), is responsible for processing non-symbolic and symbolic quantitative information in humans and many non-human animals. It allows us to form approximate representations of larger cardinal quantities without counting. ANS can thus be used for estimating the number of observed objects, or to apprehend the difference in quantity between collections of objects, as well as their symbolic representations (Dehaene 1997/2011). In addition, the ANS can give a rough sense of "more" when it comes to quantities: unlike subitizing, it is not limited to small quantities and can be seen as giving an intuition that numbers form some type of increasing progression. However, as an estimation system, ANS becomes increasingly inaccurate as the quantities become larger. Moreover, even if its acuity improves throughout development, it always retains the quasi-logarithmic character of decreasing accuracy as the quantities increase.

1.2 Symbolic number system and first concepts acquisition

It is important to note that the two core cognitive systems persist despite the possible acquisition of a symbolic number system, the Linguistic Number System (LNS). Fuson (1988), Wynn (1990), and Griffin and Case (1996), and others have reported that children first learn the list of numerals by heart, and it is only afterward that they start associating meanings with its items.

The paradigm which we work in explains a lot about the initial acquisition of conceptual content. We know that children learn number-word meanings one at a time, slowly understanding which conceptual content is quantity-specific (see Condry & Spelke 2008; Le Corre & Carey 2007; Le Corre et al. 2006; Negen & Sarnecka 2012; Sarnecka & Gelman 2004; Sarnecka & Lee 2009; Wynn 1990, 1992). In an idealised way the process goes as follows: children first know names of numbers by heart (they are *pre-number knowers*), then they start associating meanings with these names (they climb the *knower-level* ladder becoming *one*-, *two*-, *three*- and *four-knowers*). The first conceptual content is not necessarily quantity-specific, but nevertheless necessary for numerical-language acquisition. For instance, children that are at the three-knower level understand that numbers apply to discrete entities (such as blocks), but not to continuous entities (such as water) (Slusser & Sarnecka 2007).

An important conceptual change occurs when children make the leap to understanding cardinality (called *cardinality-principle knowers*, or *CP-knowers*), that is, when they can recognize that number words pick out numerosity as opposed to some other property of collections, such as total summed area or contour length. Similarly, only CP-knowers understand that moving forward one word in the number-word list means adding one item to the named collection (Fuson 1988; Gelman & Gallistel 1978; Schaeffer et al. 1974). This is an important change as the meaning of *all* number words is likely to change. The PIS-based references for "one", "two", "three" and "four" are probably at least partly altered as well, since these numbers will start being a part of a sequence. This is the first important conceptual change that occurs and it is still not clear how children become CP-knowers, and which processes lead children to understand the meanings of large numbers and of the concept of natural number in general.

1.3 Learning the mature concept through "bootstrapping"

Currently the most fruitful, explicatively empowered and, hence, the most explored theory states that grasping large exact natural numbers and the concept of exact natural number in general is possible through a process of *bootstrapping* (*e.g.* Sarnecka & Carey 2008; Carey 2009; and Spelke 2011). In addition to the systems of representation arising from core cognition, there also exist symbolic representations that develop as historical or cultural constructions, as well as in individual learning. The idea of bootstrapping originally came from Quine (1960), in which the author pictures concepts in a holistic manner as an interconnected web. Conceptual change, or meaning shift, of one of the concepts within the web induces a conceptual change of the neighboring concepts, and thus enables the creation of new concepts.

When it comes to bootstrapping related to number cognition, the existing conceptual web of representations resulting from processing quantitative information is used to create a new concept of exact natural number. Bootstrapping has been described metaphorically as similar to a ladder or a chimney. The idea behind the metaphor is that steps of the ladder and walls of the chimney must be constructed in a way that enables climbing them and reaching the top (a new value). Similarly, it is clear that the conceptual web must have certain characteristics to allow for bootstrapping. There is another aspect of this metaphor, which is important for the topic here: just as all walls will not enable successful climbing, not just any conceptual content makes it possible to bootstrap new concepts. The important question for us is why some walls meet whereas others do not – that is, what enables the fruitful bootstrapping from some combination of concepts?

Carey (2009) adapts this picture to the needs of the developmental psychology oriented toward early language acquisition. Bootstrapping in the case of natural numbers is a particular kind of enrichment of the conceptual framework related to numerosity and quantity that indirectly allows grasping the meaning of the mature concept of natural number. When it comes to natural numbers, different processes are considered as probable candidates to be involved in bootstrapping.

One possibility is that all number concepts come from the ANS. According to this alternative, the conceptual content of exact numerals emerge from or supervenes on the experience of approximating quantities with ANS (Dehaene 1997/2011; Halberda & Feigenson 2008). This possibility is supported by evidence linking ANS acuity to school math achievement. For instance, Halberda et al. (2008) found that the ANS acuity of 14-year-olds retrospectively predicted their math achievement scores back to kindergarten. However, evidence against this idea was provided by Negen and Sarnecka (2014), who found no link between children's ANS acuity and their number-knower level. It has also been shown that children from low-income backgrounds, often under-performing in math tests, differ very little from children from higher-income backgrounds in their ANS acuity, yet they can differ greatly in their knowledge of counting and the linguistic number system (Goldman et al. 2012).

A competing hypothesis states that grasping the conceptual content of number names emerges on the basis of the knowledge of number words, subitizing and understanding of the cardinality principle (Carey 2009; Izard et al. 2008; Sarnecka & Carey 2008; Sarnecka & Gelman 2004; Sarnecka & Wright 2013). According to this hypothesis, adding one element to a set correlates to progressing one step on the line of numerals, thus capturing an early idea of the successor function.

According to both hypotheses, from various unrelated conceptual inputs – either innate or learned – the child can bootstrap the idea that numerals form an endless progression of increasingly growing magnitudes. What we aim to show is that neither of them fully accounts for the expected conceptual change that involves *regular spacing* between subsequent elements. Usually criticism of one or another hypothesis is based on showing that it is invalidated by some experimental research, as when Negen and Sarnecka (2014) found no link between children's ANS acuity and their number-knower level. Our endeavour, however, will be different. We are going to proceed with conceptual analysis and explore the conceptual framework needed for such bootstrapping. Can ANS alone be enough to ensure that we become CP-knowers and thus become able to grasp the LNS? Or perhaps the key ability is to generalize over subitizing as - in a sense - claimed by proponents of the cardinality principle solution? Or perhaps we need both of these abilities, with other core cognitive systems also possibly engaging in the process.

Returning to the metaphor, the whole idea of bootstrapping relies on the idea that there exist concepts that can serve as ladder steps or chimney walls on the way up to forming new concepts. The regularity of the LNS, however, seems to be something we cannot inherit from ANS, for reasons already mentioned. First, ANS is universal whereas LNS not. Second, ANS acts only on very little fragments of what we know about the sequence. Third, even if we would agree that ANS forms a progression, this progression might be – as Dehaene recognises himself – better represented by a logarithmic progression.

Neither can the regularity of the number progression be inherited from PIS and the cardinality principle. Subitizing stops working at four objects and it does not have the potential to provide conceptual content for bigger quantities. The cardinality principle and the "add one" procedure could explain regularity only if the added element is always of the same size. But the PIS or the cardinality principle give no guarantee of that. The primitive idea of discrete quantity we receive from them does not imply that the distance between three and four is the same as that between, say, nine and ten.

2. Linguistic Number System and the number line

2.1 Regularity of Linguistic Number System

Learning the meanings of individual items and the meaning of the full progression of the Linguistic Number System is not a straightforward process and the learning process clearly has two distinct aspects, syntactical and semantical[4]. For example, research shows that children learn to recite a

[4] In this paper we will not take a radical stand relating to the question on whether the syntax can be learned purely arbitrarily, without any appeal to semantics. When it comes to semantical content, the paradigm of core cognition suggest there exists quantity-related pre-verbal semantic content.

sequence of numerals without being able to match the number words to quantities (Fuson 1988; Wynn 1990; Davidson et al. 2014). Thus, instead of providing a semantically saturated structure, initially the numeral sequence forms a so-called *placeholder* structure that is only later filled in with meaning during the learning process.

This setup is already present, at least to some extent, in the research of many cognitive psychologists. When Sarnecka (2015) explains the objectives of developmental cognitive psychologists studying natural numbers, she speaks about understanding how the names of natural numbers acquire conceptual content. Similarly, Carey (2009) draws a clear distinction between the syntactical layer of symbolic representations (formation of the placeholder structure) and the semantical layer of symbol interpretations (she speaks of object-files gradually getting conceptual content). As we understand it, the endeavor of (many) cognitive scientists working in number cognition amounts to investigating the part of conceptual content of the sequence of number names that comes from core cognition. Even more generally, they seem to concentrate on the wider issue of early language acquisition. For both approaches, the setting is clear: the sequence of names for numbers is seen as picking out a unique mathematical object and its conceptual content can be traced down to our cognitive abilities.

In the previous section we presented data relating the semantical aspect of conceptual content of natural numbers acquisition. In the end, we asked whether the crucial characteristic of regularity could be derived from them. But perhaps we should not look for regularity from the semantical side? In what follows, we want to pursue the idea that the regularity of the natural number structure, and the corresponding regular distances in the natural number line, may come rather from the syntactical side.

Let us first take a careful look at the structure of the most widespread system of notation for natural numbers: Arabic numerals. It is easy to fall under the illusion that our contemporary way of expressing them is somehow developmentally inevitable. But it is important to remember that the contemporary numeral systems, such as the one used in English, are the product of a long line of development and have some important characteristics that are not general to all numeral systems. Before it could develop in the way we know it today, there has been a great multitude of ways in which human beings have developed expressions and notations for quantities, beginning from various types of tallying (stroke notation) or systems based on body parts (Butterworth 1999, Dehaene 1997/2011). In this context, what is the main distinguishing feature of the contemporary numeral system? What tells our system apart from other systems of notations? For instance, why is it that Funes' system of numerals, described below, is not suitable for representing natural numbers as we know them today?

Funes the Memorious, a fictional character from one of the Borges novels (Borges 2000, 91-99), due to a severe brain injury happened to have an infinite working memory. At the same time, he was critically lacking all ability to systematize his knowledge. Beyond other extraordinary aspects of Funes' condition that Borges describes, Funes invented his own numerical system[5]:

> He told me that in 1886 he had invented a numbering system original with himself, and that within a very few days he had passed the twenty-four thousand mark. (…) Instead of seven thousand thirteen (7013), he would say, for instance, "Máximo Pérez"; instead of seven thousand fourteen (7014), "the railroad"; other numbers were "Luis Melián Lafinur," "Olimar," "sulfur," "clubs," "the whale," "gas," "a stewpot," "Napoleon,""Agustín de Vedia." Instead of five hundred (500), he said "nine." Every word had a particular figure attached to it, a sort of marker; the later ones were extremely complicated....

For the sake of our thought experiment, imagine a child who learns about numbers only from Funes. According to the paradigm we work in, as an infant, this child will learn an arbitrary sequence of Funes numerals the way other infants do with standard numerals. It might go something like: "Luis Melián Lafinur", "Olimar", "sulfur", "clubs", "the whale", "gas",… Through the ability to subitize, she will learn to associate the meaning of "Luis Melián Lafinur" (that might be Funes' one) to collections of one element, "Olimar" (Funes' two) to collections of two elements, "sulphur" (Funes' three) to collections of three elements and also "the reins" (Funes' four) to collections of four elements. Thanks to the training in associating number names with representations given by subitizing, the Approximate Number System, as well as the cultural context and conceptual interconnections, she will eventually understand that numerals refer to finite cardinalities. Thanks to the growing magnitudes of cardinalities, perhaps due to a sense of "more" arising from the functioning of the ANS, she will apprehend the idea that the quantities form a progression of growing magnitudes. Finally, thanks to the cardinality principle, she will eventually learn that adding one element to the set will allow her to progress one name on her number line. Funes' child will learn all that from the Funes' number line. But – we argue – this

[5] Benacerraf makes of this passage the motto of his paper "Recantation or any old omega-sequence would do after all" (1996), the follow up of the famous "What Numbers Could Not Be" (1965).

does not guarantee that the child acquires the Linguistic Number System used in the Western society. This is how Borges himself objects to Funes:

> I tried to explain to Funes that his rhapsody of unconnected words was exactly the opposite of a number system. I told him that when one said "365" one said "three hundreds, six tens, and five ones," a breakdown impossible with the "numbers" Nigger Timoteo or a ponchoful of meat. Funes either could not or would not understand me. (Borges 2000, 97)

What exactly goes wrong? As we claim above, one widely accepted characteristic of natural numbers is that they are best represented by *recursive* and in consequence, *regular*, systems of numerals. One of the features of such number system and the corresponding number line is that distances between successive elements are always the same.

Numerals from Funes' number line do not embrace the idea of equal distances between the elements. The Funes numerals are separate entities that show no internal syntactical structure. In this way, for the Funes' number line, ANS and subitizing could indeed be sufficient in order to bootstrap the structure of numbers. The contemporary Linguistic Number System, however, *does* show syntactical structure, and a child needs to learn it in order to understand the concept of natural number. The regularity of the syntactical structure of the LNS, however, is something that we cannot extract from subitizing, ANS and the cardinality principle. Some further factor is needed.

2.2 Linguistic Number Line

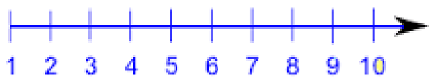
As argued in the previous subsection, natural numbers are best represented by a *recursive* system of numerals rather than any sequence of random names. The *regularity* of the system, the equality of distances, is the consequence of the regularity of the successor operation. Funes could learn his system of numerals without ever forming a recursive sequence, or being able to generate new number names, because he had an infinite memory. Borges does not tell us whether Funes was able to associate his number words with desired cardinalities, but clearly with our finite memories, none of us would accept Funes' system as adequate symbolic representation of natural numbers.

While languages differ greatly in their numeral systems, at some point contemporary languages generally show both regularity and recursivity. It is visible already in the number names, which clearly show recursivity: twenty, twenty-one, twenty-two, and so on, thirty, thirty-one, thirty-two, and so on.

Visually, regularity is represented as equal spaces between the numbers, as seen in the natural number line. In our culture everyone is familiar with the number line, and thus it can provide us with a fruitful framework to study the cognitive processes involved. In the early stages of education, children often learn basic arithmetic with the help of the number line, which is a simple visual presentation of the Linguistic Number System. It allows children to quickly determine which of two numbers is bigger, as well as to learn the basic arithmetical operations of addition and subtraction.

This regularity of the number line may seem obvious, but in fact goes against the natural tendency to place numbers on a line. Studies show that subjects from many non-arithmetical tribes put quantities on the line in a quasi-logarithmic manner (Dehaene et al. 2008).[6] Hence the distance on the number line between 1 and 2 is not the same as the one between, say, 10 and 11, corresponding to the way the ANS works. However, while the regularity of the natural number line goes against the ANS, it clearly facilitates the learning of meanings of the elements forming the Linguistic Number System.

LNS-generated representations have three key characteristics.[7] First, there is a unique symbol or name for each number. Second, the move from one natural number to the next is always the same, *i.e.* there are no numbers between the two numbers, and the next number is not the successor of any other natural number. Third, for every natural number, there is always a next number, making the progression of natural numbers unending.[8]



---

[6] If they put quantities on the line at all. Recent studies show that even the idea of placing quantities on a number line is foreign to some cultures (Núñez 2011).
[7] Here we speak about LNS in general, not about the contemporary recursive progression.
[8] It should be made clear that historically, the number line has not always been a mere visual representation of natural numbers. In Euclid's *Elements*, for example, natural numbers were primarily represented as line segments.

In the natural number line, all aspects are present. Clearly a finite image can never contain the whole number line, but it can convey sufficient information to evoke the natural number structure. Every notch in the natural number line is named by a unique number symbol. The distances between the notches are regular, thus visualizing the idea that the successor relation is always the same. And finally, the direction of the arrow informs the learner that the line forms a progression that could be continued indefinitely.[9]

With its enduring success in early education (Gravemeijer 2014), it seems plausible that grasping the natural number line somehow involves using the existing abilities with quantities. Indeed, it appears to carry features rising both from subitizing and the ANS. The first four numbers can be seen to come from PIS and subitizing. Similarly, the direction of the arrow can be seen as capturing the sense of "more" arising from the ANS. Regularity, however, is something that the core systems of PIS and ANS cannot sufficiently explain. Presenting LNS with the help of the number line makes this obvious. In the visual representation of the progression of quantities, there is no basic need to have regular spaces between the numbers, yet the idea of regularity is crucial to LNS and children learning arithmetic. The sensible hypothesis is that regularity comes from some *other* cognitive faculty.

However, before that we should ask what is happening conceptually when grasping the number line and the LNS? Clearly grasping the natural number line connects to the general question of grasping "large" natural numbers.[10] Since subitizing only works for small quantities and the ANS does not give discrete numbers, this is an important developmental question. Somehow children grasp *new* number concepts with the help of the number line. When they extract the idea of the successor function from a multitude of cognitive factors (subitizing, ANS, sensitivity to regularity), they get tools for grasping larger and larger natural numbers. But what exactly is the mechanism that allows children to grasp new concepts such as large natural numbers in the first place? It is this general question of concept-acquisition we will move onto next.

---

[9] These properties have of course been presented well before the modern number line was developed. In fact, the oldest remaining artefacts for keeping track of quantities show a use of the principles of progression and regularity. In systems of tallying (or stroke notation), one notch on wood or bone signified a move to the next number. The famous Ishango Bone – more than 20,000 years old – shows this kind of regular progression of numerosities (Bogoshi et al. 1987).

[10] Here "large" is not meant to have a strict meaning and may as well refer to 10 as 1000.

3 Fodor's challenge

In a wider philosophical context, bootstrapping is a plausible answer to a larger problem of concept acquisition. It allows us to overcome the so-called “learning paradox” and thus explain how learning new concepts is possible. The paradox is as old as philosophy itself and it has received a multitude of different solutions. Plato, for example, solved it by postulating an existence of gods who inserted the wisdom to human souls. In the most recent times an influential proposal has been formulated by Noam Chomsky and his idea of genetically determined deep structure of language that gets activated by experience (Chomsky 2002).

The paradox has received its most famous formulation by Fodor who said that:

> it is never possible to learn a richer logic on the basis of a weaker logic, if what you mean by learning is hypothesis formation and confirmation… There literally isn’t such a thing as the notion of learning a conceptual system richer than the one that one already has; we simply have no idea of what it would be like to get from a conceptually impoverished to a conceptually richer system by anything like a process of learning. (Fodor 1980, p. 148-149)

Carey (2009) explicitly formulates her bootstrapping theory as an alternative to Fodor. Since we are working within the paradigm that bootstrapping belongs to, we should spend some time taking a closer look at what Fodor proposed.

Fodor’s solution to the paradox comes from two factors. The first is *radical nativism*, the position that all primitive concepts are innate, and complex concepts are formed out of primitive ones (Fodor 1975, 1980, 2008). The second is the *continuity thesis*, which states that the representational resources needed to express all concepts have to be available throughout development, even at the very beginning (Fodor 1975, 1980, 2008; Macnamara 1982; Pinker 1984, 1994). The only way we can acquire new concepts, Fodor argues, is by composing them from “primitive” concepts that we already have. In this way, it is possible to acquire the complex concept of, say, “white cat”, given that we possess the primitive concepts of “white” and “cat”.

This of course prompts the question how we come to acquire new concepts *at all*. Fodor overcomes this issue of novelty by proposing that learning happens through testing hypotheses.

We generate hypotheses about what a new concept could mean in terms of concepts that we already have, and then we test those hypotheses by experimenting with the external world. Children may, for example, form a hypothesis that when a rock is thrown in the air, it comes down. When they test the hypothesis in practice, they receive confirmation of it. However, if they are not familiar with the concepts "rock", "air", "throwing" or "down", there is no way to formulate the relevant hypothesis.

When it comes to the acquisition of the concept of exact natural number, the continuity thesis provides an interesting challenge, because natural numbers are abstract entities and cannot be observed. If natural numbers are considered to be primitive concepts, we would have to possess them already at birth because there is no ostensive way to acquire their first meanings. Obviously, we cannot in our finite brains possess *every* natural number concept. Indeed, the innateness claim seems unfeasible even for large finite numbers. Could we really have an innate representation for the number, say, 5,658,343? Even if some of the content of number concepts were innate, this cannot generally be the case.

If natural numbers cannot be primitive concepts, could they be composite concepts? However, it is not easy to determine what this could mean. Would that be a simple case as "white cat" or rather one that causes troubles to Fodor's theory, such as "carburetor" which is unlikely to be composite of innate primitive concepts (Steinton & Viger 2000, p. 142)?

The only option seems to be that natural numbers are composite concepts, but rather than being composed of primitive concepts, they are obtained with some mechanism that allows us to generalize on number concepts. Whatever the exact mechanism may be, it must amount to claiming that the existence of some kind of innate finite representations gives access to the infinite natural number structure. At this point, however, we are clearly pushing the limits of the continuity thesis. If we acquire a concept of an infinite structure based on a small finite collection of discrete quantities, are we not in fact acquiring a genuinely *new* concept? If we want to insist on the continuity thesis, we must find a way to explain the infinite natural number structure as being composed of primitive concepts. Perhaps that line of research can be pursued but, at this point, it appears that we are stretching the idea of continuity too much.

The idea that numbers are composite concepts received support from many researchers who believe that we have an innate cognitive system to represent exact numbers. In addition to the paradigm we are referring to in this paper, worth noting are the proposals of Leslie et al. (2007) and of Gelman and Butterworth (2005).[11] We believe that bootstrapping gives a more plausible framework for grasping such principles, but the theory here is not tied to a single paradigm. The important point is that the exact natural number concepts are derived by grasping a general principle about discrete quantities, not by composing them from primitive concepts.

Regardless of whether we agree with the details of Carey's account, it is clear that bootstrapping has several advantages as a framework for theories on how we acquire the exact natural number concept. First, it makes use of core cognitive systems responsible for processing quantitative information – in Carey's account these are the approximate number system and subitizing – which we know to be present already in infancy. Second, it does not introduce a new type of cognitive mechanism. In the philosophy of mathematics, access to mathematical objects has often been explained by appeal to mathematical intuition. While we have argued that the present explanations of bootstrapping leave open questions, one clear strength of them is the lack of such intractable epistemological mechanisms. Third, and importantly, with bootstrapping we avoid the position that number concepts of the Linguistic Number System are innate in our brain. The severity of the contrary position cannot be overstated. If an infinite system has a representation in the brain, it has to be due to some mechanism that allows the conceptual move from finite to infinite.[12] The finiteness of our brain cannot be questioned, and as such the position that all natural numbers have innate representations in the brain must be simply false.

However, bootstrapping should not be treated as a magical solution to the problem. The idea that natural numbers are understood as based on an ability to bootstrap from fragmentary conceptual information acquired through core cognitive systems is appealing. But as we have specified above, it should not be forgotten that there are – in principle – various ways of executing such bootstrapping. It is not clear that there exists a mechanism that allows us to bootstrap a new

[11] Leslie et al. (2007) suggested that instead of having innate representations of exact numbers, humans are equipped with an innate capacity to grasp principles from which these representations can be derived, such as exact equality or the successor function. Gelman and Butterworth (2005) have argued for the position that we can have a non-linguistic representation of natural numbers. Based on the Oksapmin people of New Guinea, they argue that we have a non-linguistic representation of natural numbers. The Oksapmin do not have number words, but rather use body positions to signify different quantities. However, when they entered plantations to work, the Oksapmin were quick to learn counting rules and the numeral vocabulary. Gelman and Butterworth argue that this could not have been possible without a prior non-linguistic representation of natural numbers.

[12] See Pantsar (2015) for an account of how the move from finite to infinite can be conceptually explained in terms of metaphorical thinking in mathematics.

concept from one set of concepts, but not from some other set. We still do not know how it is possible to distinguish “fruitful” sets of concepts from the “random” ones (the walls of the chimney and steps of the ladder that enable climbing). Finally, if from a different set of concepts we can bootstrap a different concept, how can we determine which is the operating one?

Possible counter-examples to the inevitability of the bootstrapping process have been proposed in the literature. For example, Rips et al. (2006) have argued that there is nothing in the bootstrapping process to prevent *loops* in the number line. The resulting numeral system could be modulo 20, for example, or follow a structure in which a numeral can be the successor of several numerals. In any case, the number system would not be like natural number structure known in our culture.

It is an empirical question whether people show a tendency toward such loops in the number line, and we must be careful not to simply rule out that possibility.[13] However, studies concerning the number line, with both children and non-arithmetical tribes, do not support the view that there is a natural tendency toward such loops in the numeral system (Dehaene et al. 2008). One of the most general understandings people have of the graphical representation of the natural numbers’ progression (linear or quasi-logarithmic) is that it is seen to continue indefinitely in the same direction.[14]

Perhaps the lack of loops can be explained by the ANS: while the ability to determine quantities becomes less precise as the quantities become bigger, it does not support the existence of loops in the system. Thus the above conception of ANS and subitizing as the cognitive basis for bootstrapping suggests a plausible solution as to why the number line is perceived as moving in one direction. However, our purpose here is not to establish that such alternative bootstrapping processes are impossible. Rather, our endeavour is to suggest that there may be additional conceptual content based on basic cognitive abilities, which superimposes on the ones accepted in the current accounts of the bootstrapping process. Even if we were able to explain why the natural number line has no loops, we still need to answer why the spaces between numbers on the line are regular. If our knowledge of natural numbers is based on ANS, why do we not follow the quasi-logarithmic characteristic of it?

[13] The authors of this paper are currently preparing an experiment to test this.
[14] Keeping in mind the qualification presented in Núñez (2011) that not all cultures appear to have any conception at all of numerosities as forming a line.

4 Sensitivity to rhythm

We have seen that while subitizing (together with the cardinality principle) and the Approximate Number System provide a plausible framework for the bootstrapping process, they are not enough to grasp the idea of the discrete natural number structure (the Linguistic Number System) and the corresponding number line. One main characteristic of the ANS, often seen as providing the general idea that quantities form an indefinitely continuing progression, is that, unlike the number line, it does not form regular intervals between quantities. Hence, the essence of the natural number line is that numbers form a regular progression. Any part of the number line behaves the same as any other part of the same size, *i.e.,* the progression 4-5-6-7 has the same intervals as, say, the progression 4555-4556-4557-4558. With the ANS, this is not the case: the ability to distinguish between approximate numerosities weakens as they become larger. The borders between numbers become blurred, so to speak, which does not support discreteness of the elements.

This problem is perhaps partly solved by the ability to subitize. Together with the cardinality principle, it gives us a small group of discrete numerosities, which can be seen as having regular intervals to generalise upon (Pantsar 2014). Thus the idea of combined influence of subitizing and the ANS in giving us a regular progression of discrete natural numbers seems plausible. At this point, however, we must ask one crucial question: if subitizing indeed gives us the idea of regular, discrete structure, why do we so naturally follow it rather than the quasi-logarithmic structure of the ANS when we move further on along the number line, i.e. when we bootstrap the natural number concept?

Moeller et al. (2009) established that young children tend to see the number line in a logarithmic manner early on, but then gradually switch to a linear one. The natural tendency for logarithmic number lines has also been detected in non-arithmetical tribes (Dehaene et al. 2008), as well as being observed with people experiencing number synesthesia. However, this natural tendency is - in the process of learning - replaced by a progression of numbers with regular intervals. In the bootstrapping theory based on ANS and subitizing, this appears to be left unexplained.

Here we want to suggest an answer: in addition to the representations generated by the two core cognitive systems that are studied in the context of natural number concept formation, there is another primitive ability that plays a key role in bootstrapping. This ability is sensitivity to *rhythm* (or beat induction), which empirical studies have found to be well established in young children. As of yet, the sensitivity to rhythm has not been applied in explanations of the cognitive basis of natural numbers. This is surprising, because sensitivity to rhythm had obvious connections to regularity. Rhythm, by definition, refers to a regular succession of elements. Since the regularity of the natural number line remains unexplained, the tendency to regularity through rhythm deserves our attention.

Already at two days old, humans manifest sensitivity to rhythm (Zenter and Eerola 2010; Winkler 2009; Honing 2012). This sensitivity appears to play a key role in the development of linguistic ability. The study of Dellatolas et al. (2009), for instance, shows that an ability for rhythm reproduction already in kindergarten age is predictive of later reading performance. Przybylski et al. (2013) also show a connection between recognizing rhythmic patterns and grasping the structure of language.

Rhythmic patterns are regular, and as such they can be characterized by recursive processes. Just as the successor function of natural numbers can be characterized recursively as "plus one" (or equivalent), rhythmic patterns can be characterized recursively as repeating the previous steps. Aside from rhythm and mathematics, such repeatable, recursive processes are much discussed in the literature on linguistic development. According to Chomsky (2002), the deep structure of human languages is determined by the recursivity of the syntax. Indeed, Hauser, Chomsky and Fitch (2002) have famously argued that recursion, i.e. the lack of upper bounds in the length of syntactical constructions, is the only uniquely human part of our languages.[15]

There is little doubt about the existence of the sensitivity to rhythm already in very young children. It also seems widely accepted that language is structured in a recursive and regular manner. The correlation between the two also seems highly plausible. Languages have characteristic rhythms and there are numerous ways in which the natural connection between language and rhythm is easily seen to exist, from the rhythmic chanting of many tribes to the common use of songs in teaching language to children. Since the connection between rhythm and language is so ubiquitous, our next question is how rhythm and the learning of *numbers* are connected.

[15] The idea of recursion as a universal characteristic of human languages was challenged by Everett (2005) on the basis of the Pirahã language, which, according to him, shows no recursion. Interestingly, the Pirahã language also has no numerical system beyond one, two and "many".

Above we established the need to answer why the visual presentation of natural numbers as a number line follows regular intervals instead of the quasi-logarithmic intervals of the ANS. Under the hypothesis that rhythm and regularity are key components in learning languages, a possible answer presents itself. Regularity is important for languages, but the number line is *all* about regularity.

Could this be the missing piece in the bootstrapping puzzle? The discrete quantities given by the ability to subitize arise from utilizing the sensitivity to regularity that we have. In this way, the hypothesis is that in addition to ANS and subitizing, also regularity plays a role and enables grasping natural numbers as forming a regular progression of discrete objects, spatially represented by the regular number line.

While these connections between rhythm, regularity and the number line have not been explicitly studied, they are related to a well-known discussion. The recursiveness of the natural number structure is often explained as following from the recursive structure of the language (*e.g.* Bloom 2000; Maddy 2014). Indeed, the way we can add more words or syllables to phrases in general seems to closely correlate with creating and learning new numerals. Numeral systems in many languages are syntactically built to follow the idea that numerals form a progression that can be continued indefinitely. As such it facilitates the teaching of this feature to new generations, thus leading to what Menary (2015) calls "enculturated" arithmetical cognition.

The relation between the recursive structure of the language and recursivity of numerals is difficult to establish. It is not possible to know whether numerals developed recursive character because the languages are recursively structured, or whether the development was the other way around: because the pre-verbal conception of natural numbers included the idea of regular progression, numerals were developed to express this idea. It could also be that the sensitivity to rhythm and regularity influences both the linguistic and numerical abilities (at least partly) independently. For our purposes, however, it makes little difference. We want to argue that sensitivity to rhythm plays a role in grasping the regularity of the Linguistic Number System and the corresponding number line. How this is connected to linguistic ability is a highly interesting question, but as long as we must amend the bootstrapping theory by including sensitivity to rhythm somehow, the exact nature of the mechanism does not change the argument.

Thus the sensitivity to rhythm can be the missing piece in bootstrapping, giving us the following picture of grasping the natural number structure:

1. subitizing gives us the concept of discrete quantity;

2. the ANS gives the idea that numerals form a progression;

3. finally, the sensitivity to rhythm ensures that the progression is regular.

This model would explain many of the problems we have in explaining bootstrapping. First, it would explain why we bootstrap the discrete system instead of a logarithmic one: we can have a pre-linguistic conception of the regularity of natural numbers.[16] Second, it enables us to hypothesize that the common recursive grammatical structure of language and the natural number progression are determined at least partially by this sensitivity to regular progression. Third, it would explain the occurrences of regularity when it comes to symbolic representations of quantities that do not belong to natural language, such as stroke or binary notations.

The last point relates to symbolic representations. In addition to using body parts for counting, the earliest forms of keeping track of quantities are various types of tallying, in which adding one stroke corresponds to the successor principle (Dehaene 1997/2011, 80-83). The stroke notation continues to be used parallel to the linguistic recursive number system, and it is plausible that tallying was developed to have ways to communicate the idea of the regular progression of numbers. This is a central point because, essentially, everything there is to know about the natural number structure can be present in a system of tallying. Indeed, in a system of tallying, the essence of the structure is present in a much clearer form than in any natural language, which starts showing the recursive structure usually at quite a late stage. With tallying, unlike in English, for example, counting from one to ten immediately shows regularity and recursivity of the successor operation. In other words, systems of tallying enable grasping the successor principle in a very pure form. Obviously tallying gets replaced by other systems of notation in most everyday contexts, because arithmetical operations, even addition, quickly becomes very time and space consuming.

[16] Or at least one that precedes being proficient with numerals.

Tallying has often been preceded by other ways of keeping track of quantities, e.g. with fingers and other body parts. Extending fingers, for example, is generally one of the first ways that western children learn natural numbers. But unlike in tallying, in extending fingers all stages are not the same: there are obvious differences in the fingers themselves. So how does the child grasp that counting is not about the fingers, but rather about numerals? Fingers are obviously used to represent quantities. But when counting with fingers, the crucial developmental part is the *process*, not the particular quantities.

How is this process possible? The answer we have wanted to suggest here is that it is due to children having a natural sense of rhythm and regularity. Even though the fingers are different, the children realize that every stage in the process is similar, i.e., they grasp the *regularity* of the process. By understanding regularity, they can grasp the successor principle and, eventually, the natural number structure. This is why the interpretation of the cardinality principle, defined very broadly as understanding that adding one element to the set means to move one item forward in the numeral sequence, is regular. Without being explicitly told, we interpret the situation where added elements are all similar.[17]

In the above argument, we do not want to claim that sensitivity to rhythm and regularity is a core cognitive ability comparable to the Parallel Individuation System or the Approximate Number System. Indeed, while there is already abundant data showing how prevalent the sensitivity to rhythm is, there does not seem to be any consensus on what innate ability is responsible for it and how it relates to our cognitive capacities. We are also conscious of Piaget's (1965) proposal that sensitivity to regularities and symmetries is obtained through the process of empirical experience. Furthermore, we do not want to claim that incorporating sensitivity to rhythm into the research paradigm will immediately solve all the problems in number cognition. It will take a great deal of work before enough is known to reveal the physiological phenomena behind the sensitivity to rhythm, not to mention the cognitive architecture that employs this in grasping natural numbers.

Instead of making such claims, in the spirit of narrowing the gap between philosophers and empirical researchers of mathematical cognition, the purpose of this paper has been to propose a hypothesis that opens up new possibilities for future studies, both empirical and philosophical.

[17] Our investigations could be further reformulated in such a way that it captures the interdependence between the ordinal and the cardinal interpretation of natural number. Izard et al. 2008 provides preliminary input to this topic. We believe a separate paper should be devoted to this issue.

However, even with the sizable gaps in our current knowledge of the topic, the hypothesis is firmly grounded in empirical studies. In addition to drawing from the empirical data on number cognition, the data on sensitivity to rhythm clearly points to its importance for cognitive processes.

While lacking in direct empirical evidence on number cognition, the hypothesis about sensitivity to rhythm receives support from a neighboring field: language cognition. In that paradigm the sensitivity to regularity has been used to acquire important results (*e.g.* Dellatolas et al. 2009, Przybylski et al. 2013). Given the already widely accepted connection between numerical cognition and linguistic cognition, one would expect the sensitivity to rhythm to receive more attention in the future, also when it comes to the foundations of arithmetic. It will obviously take a great deal of empirical work before the connection between rhythm and bootstrapping of the exact natural number concept can be explained. But, on a theoretical level, that connection carries a lot of potential for future research.

Conclusions

The main objective of this paper has been to argue that there is an important feature of the natural number progression that cannot be explained by the current theories of bootstrapping: namely, the regularity and recursivity of the progression of increasingly growing magnitudes. An equally important aspect of the paper, however, is how we have arrived at that argument. While every effort has been made to be as empirically informed as possible, the argument was formulated by using philosophical tools, such as conceptual analysis. Regardless of whether one ultimately accepts the argumentation in this paper, we contend that philosophical tools can, in general, be highly useful when explicating the characteristics of the exact natural number system in terms of numerical cognition.

Our endeavour has been to investigate the conceptual content that a syntactic placeholder structure (the sequence of numerals) gets from the core cognitive systems, social interactions, and personal concept forming. We believe that this kind of research can be fruitful both for cognitive science and philosophy. However, it requires clear and coherent use of the key concepts. Too often, empirical researchers are less than careful in this. To show just one example of conceptual confusion, see the following quotation:

> Humans possess two nonverbal systems capable of representing numbers, both limited in their representational power: the first one represents numbers in an approximate fashion, and the second one conveys information about small numbers only. (Izard et al. 2008, 491)

For the sake of conceptual clarity, we propose that this should be reformulated as follows:

> Humans possess two nonverbal systems *which enable humans to react appropriately to quantitative information and provide conceptual content for numerical expressions*, both limited in their representational power: the first one *provides content of quantity* in an approximate fashion, and the second one conveys information about small *quantities* only.

Without sufficient conceptual clarity, expressions such as “non-verbal representations” or “numbers” become vague, which makes further analysis difficult. When in the quotation above Izard and her collaborators use the expression “number”, it is not in the sense of “abstract natural number”, but rather as the quantitative information provided by core cognition. We believe that a distinction between the two is crucial for further progress in the field of number cognition and its philosophy. We have also avoided controversial concepts such as references to non-verbal representations, which are accepted in certain accounts but not in others.

This perspective fits well with the bootstrapping paradigm and, while that is the paradigm we explore in this paper, we believe it can also be used in analysing the expressive power of other paradigms. Essentially, we reformulate the two stages of bootstrapping: we treat a formulation of a *placeholder structure* as calling upon the syntactical level of natural number concept, and the *interpretation* stage is understood as filling in the conceptual content into object files. In this way we make Carey’s perspective slightly more rigid, but we do that to create a model that makes it possible to understand how different aspects of the bootstrapping process function.

As a result of this perspective, we have observed that there is a feature of the contemporary Linguistic Number System whose conceptual content cannot be triggered (in the bootstrapping sense) by any recognised resource either in core cognition or in symbolic language: the regularity of the number line or the recursivity of the successor function. In doing this, we point to a possible source in the research on number cognition that has not been taken into account until now. In consequence, similar to theoretical models offering predictions on further behaviour of the items they describe, thanks to conceptual engineering we offer a way to fill in the gap in the conceptual content necessary for bootstrapping the mature concept of natural number.

References

Benacerraf, P. (1965), What Numbers Could not Be, *The Philosophical Review*, (74)1, 47-73.

Benacerraf, P. (1996), Recantation *or* Any old ω-sequence would do after all, *Philosophia Mathematica* (3)4, 184–189.

Bloom, P. (2000), *How children learn the meanings of words,* Cambridge, MA: MIT Press.

Bogoshi, J., Naidoo, K. and Webb, J.: "The oldest mathematical artifact", *Math. Gazette*, 71:458 (1987) 294.

Borges, J.L. (2000), Funes, His Memory, In: *Fictions,* 91-99, Penguin Books.

Butterworth, B. (1999), *What Counts: How Every Brain Is Hardwired for Math*, New York: Free Press.

Carey, S. (2009), *The Origin of Concepts,* New York: Oxford University Press.

Chomsky, N. (2002). *Syntactic structures*. Walter de Gruyter.

Condry, K. F., & Spelke, E. S. (2008), The development of language and abstract concepts: The case of natural number, *Journal of Experimental Psychology: General*, *137*(1), 22–38.

Davidson, K., Eng, K. & Barner, D. 2012, “Does learning to count involve a semantic induction?”, *Cognition* 123: 162-173.

Dehaene, S (1997/2011), *The number sense: How the mind creates mathematics* (second edition 2011), New York: Oxford University Press.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures, *Science*, 320(5880), 1217-1220.

Dellatolas, G., Watier, L., Le Normand, M. T., Lubart, T., & Chevrie-Muller, C. (2009). Rhythm reproduction in kindergarten, reading performance at second grade, and developmental dyslexia theories. *Archives of Clinical Neuropsychology*, *24*(6), 555-563.

Everett, D. (2005). Cultural constraints on grammar and cognition in Pirahã. *Current anthropology*, *46*(4), 621-646.

Feigenson, et al. (2004), Core systems of number, *Trends in Cog. Sci.*, 8(7), 307–314.

Fodor, J. (1975). *The Language of Thought.* New York: Harvard University Press.

Fodor J. (1980). Fixation of belief and concept acquisition. In: Massimo Piattelli-Palmarini (ed.), *Language and Learning: The Debate Between Jean Piaget and Noam Chomsky*. Harvard University Press: 142--149.

Fodor J. (2008). *Lot 2: The Language of Thought Revisited.* Oxford University Press.

Fuson, K. C. (1988). *Children's counting and concepts of number.* New York, NY, US: Springer-Verlag Publishing.

Gelman, R., & Butterworth, B. (2005), Number and language: how are they related?. *Trends in cognitive sciences,* 9(1), 6-10.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number.* Cambridge, Mass.: Harvard University Press.

Goldman, M. C., Anaya, T. T. D., Negen, J., Flores, C., Alcala, C., Guzman, B., … & Sarnecka, B. W. (2012, June). *Bilingual preschoolers' early math skills in Spanish and English*. Presented at the Head Start National Research Conference, Washington D.C.

Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*, 496–499.

Gravemeijer, K. (2014). Number lines in mathematics education, in Lerman, S.: *Encyclopedia of Mathematics Education*, Springer: 466-270.

Griffin, S., & Case, R. (1996). Evaluating the breadth and depth of training effects, when central conceptual structures are taught. *Monographs of the Society for Research in Child Development*, 61 (1/2, Serial No. 246), 83-102.

Halberda, J. & Feigenson, L. (2008), Set representations required for acquisition of the natural number concept, *Behavioral and Brain Sciences,* 31(6), 655-656.

Halberda, J., Mazzocco, M. M. M., & Feigenson, L. (2008), Individual differences in non-verbal number acuity correlate with maths achievement, *Nature, 455,* 665–668.

Hauser, M.D., Chomsky, N. & Fitch W.T. (2002), The faculty of language: what is it, who has it, and how did it evolve?, *Science,* 298 (5598): 1569-1579.

Honing H. (2012), Without it no music: beat induction as a fundamental musical trait, *Ann. N.Y. Acad. Sci.,* 1252: 85–91.

Hyde, D. C. (2011), Two systems of non-symbolic numerical cognition, *Front. Hum. Neurosci.,* 5(150), 10-3389.

Izard, V., Pica, P., Spelke, E. S., & Dehaene, S. (2008). Exact Equality and Successor Function: Two Key Concepts on the Path towards Understanding Exact Numbers. *Philosophical Psychology*, 21(4), 491–505.

Klein, A., & Starkey, P. (1988). Universals in the development of early arithmetic cognition. *New Directions for Child and Adolescent Development*, *1988*(41), 5-26.

Le Corre, M., & Carey, S. (2007), One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles, *Cognition*, *105*(2), 395–438.

Le Corre, M., Van de Walle, G., Brannon, E. M., & Carey, S. (2006), Re-visiting the competence/performance debate in the acquisition of the counting principles, *Cognitive Psychology*, *52*(2), 130–169.

Leslie, A. M., Gallistel, C. R., & Gelman, R. (2007). Where integers come from. In P. Carruthers, S. Laurence, S. Stitch (Eds.), The innate mind: Foundations and futures (pp. 190–138). Oxford: Oxford University Press.

Macnamara, J. (1986). *Border dispute: The place of logic in psychology*. Cambridge, MA: MIT Press.

Maddy, P. (2014), A Second philosophy of arithmetic, *The Review of Symbolic Logic,* 7 (2), 222-249.

Menary, R. (2015). Mathematical cognition: A case of enculturation. In Open MIND. Open MIND. Frankfurt am Main: MIND Group.

Moeller, K., Pixner, S., Kaufmann, L., & Nuerk, H. C. (2009), Children's early mental number line: Logarithmic or decomposed linear?. *Journal of experimental child psychology,* 103(4), 503-515.

Negen, J., & Sarnecka, B. W. (2012). Number-Concept Acquisition and General Vocabulary Development. *Child Development*, *83*(6), 2019–2027.

Negen, J., & Sarnecka, B. W. (2014), Is there really a link between exact-number knowledge and approximate number system acuity in young children? *The British Journal of Developmental Psychology*.

Nieder, A. (2011), The Neural Code for Number, In: Dehaene & Brannon (eds.): *Space, Time and Number in the Brain*, London: Academic Press 2011: 107-22.

Núñez, R. E. (2011). No innate number line in the human brain. *Journal of Cross-Cultural Psychology*, *42*(4), 651-668.

Pantsar, M. (2014), An empirically feasible approach to the epistemology of arithmetic, *Synthese*, 191(17), 4201-4229.

Pantsar, M. (2015), In search of ℵ0: how infinity can be created, *Synthese,* 192(8), 2489-2511.

Piaget, J. (1965). The child's conception of number. New York: W. Norton Company & Inc.

Pinker, S. (1984), *Language Learnability and Language Development,* Cambridge, MA: Harvard University Press.

Pinker, S. (1994), *The language instinct,* New York: William Morrow.

Przybylski, L., Krifi-Papoz, S., Roch, D., Kotz, S.A., Bedoin, N., Herbillon, V., Léculier, L., Tillmann, B., (2013), Rhythmic auditory stimulation influences syntactic processing in children with developmental language disorders, *Neuropsychology* 27.1 (2013): 121.

Quine (1960), *Word and Object*, Cambridge, MA: MIT Press.

Quinon, P. (2014), From Computability over Strings of Characters to Natural Numbers. In: Olszewski, A., Brożek, B., Urbańczyk, P., (eds.), *Church's Thesis, Logic, Mind and Nature,* Copernicus Center Press, 227-270.

Rips, L. J., Asmuth, J., & Bloomfield, A. (2006). Giving the boot to the bootstrap: How not to learn the natural numbers. Cognition, 101(3), B51-B60.

Rips, L.J., Bloomfield, A. & Asmuth, J. (2008). From numerical concepts to concepts of number. Behavioral and Brain Sciences, 31(6), 623-642.

Sarnecka, B.W. (2015), Learning to Represent Exact Numbers, *Synthese,* 1-18. Published online 27 August 2015, In: Antonutti, M. & Quinon, P., *Intentionality in Mathematics,* Synthese Special Issue.

Sarnecka B.W. & Carey S. (2008), How counting represents number: What children must learn and when they learn it, *Cognition,* 108(3), 662–674.

Sarnecka B.W. & Gelman S.A. (2004), Six does not just mean a lot: preschoolers see number words as specific, *Cognition,* 92(3), 329–352.

Sarnecka, B. W., & Lee, M. D. (2009), Levels of number knowledge during early childhood, *Journal of Experimental Child Psychology*, *103*(3), 325–337.

Sarnecka B.W. & Wright C.E. (2013), The Idea of an Exact Number: Children's Understanding of Cardinality and Equinumerosity, *Cognitive Science,* 37(8):1493-1506.

Schaeffer, B., Eggleston, V. H., & Scott, J. L. (1974), Number development in young children, *Cognitive Psychology,* 6(3), 357–379.

Spelke, E. (2000), Core knowledge. *American Psychologist, 55*, 1233-1243.

Spelke, E. (2003), What makes us smart? Core knowledge and natural language. In D. Gentner and S. Goldin-Meadow (Eds.), *Language in Mind: Advances in the Investigation of Language and Thought.* Cambridge, MA: MIT Press.

Spelke, E. (2011), Natural Number and Natural Geometry, in Dehaene & Brannon (eds.): Space, Time and Number in the Brain, London: Academic Press 2011: 287-318.

Starkey, R, & Cooper, R. (1980), Perception of numbers by human infants, *Science,* 210, 1033-1035.

Stainton, R. J., & Viger, C. (2000), Jerry A. Fodor, Concepts: Where Cognitive Science Went Wrong, *Synthese,* 123(1), 131-151.

Trick, L., & Pylyshyn, Z. W. (1994), Why are small and large numbers enumerated differently? A limited capacity preattentive stage in vision, *Psychological Review,* 101, 80-102.

Winkler, I., G. Haden, O. Ladinig, et al. (2009), Newborn infants detect the beat in music, *Proc. Natl. Acad. Sci. USA*, 106: 2468–2471.

Wynn, K. (1990), Children's understanding of counting. *Cognition,* 36(2), 155–193.

Wynn, K. (1992), Children's acquisition of the number words and the counting system. *Cognitive Psychology,* 24(2), 220–251.

Zentner, M. & Eerola, T. (2010), Rhythmic engagement with music in infancy*, PNAS,* 107 (13): 5768–5773.