Magnitude and Number Sensitivity of Approximate Number System in Conceptual Spaces

1. Introduction

In (2017) Leibovich *et al*. argued that the system processing quantitative information, at least at the earliest stages of human development, does not deal with numerosity of discrete entities – as does Approximate Number System (Dehaene 1997/2011) – but with magnitude of space covered by these entities. Leibovitz *et al.* argument builds upon the idea that the perception of infants, who are able to process quantitative information, is not acute enough to distinguish small individual objects (Banks 1980, Dobson & Teller 1978). Additionally, infants younger than 5 months old are not able to individuate objects and individuation is necessary for all numerical quantity assessments (Carey 2001, see also Feigenson *et al.* 2002). In consequence, humans must be equipped with an Analogue Magnitude Representations system taking as an input a surface covered by a collection of discrete objects and returning as an output an analogue representation encoding information about the magnitude of this surface. There might exist a completely new system that is further replaced by ANS, but there also might be the case there is no ANS, but a separate AMRs system that processes magnitude of space covered by discrete entities and not the numerosity of these entities. Indeed, it is very complicated to extract a strictly numerical aspect of ANS when it gets activated by a non-symbolic input (*e.g.*, a collection of dots distributed at some surface). Numerosity is in this case strongly correlated with other dimensions such as size of surface, length of contour, density etc. Starkey & Cooper (1980) introduce this problem in the following way:

> The bases of discriminations among large values are not known and may vary with different types of arrays (for example, dot size, the spatial, distance separating adjacent dots, or surface area of the total array). The ability underlying these discriminations might be the same as that which allows adults to estimate the more numerous of two arrays differing substantially in number. (Starkey & Cooper, 1980 p. 1033).

For instance, Mix, Huttenlocher, & Levine (2002a) claim that processing of numerosity by ANS is just a byproduct of processing continuous input:

> Contrary to what has been claimed in current models of early quantification, we failed to find evidence that discrete number is represented in infancy. Instead, there is strong evidence that infants view spatial quantities in terms of total amount of substance. There are also indications that infants view sequential quantities in terms of temporal cues, such as rate, duration, and rhythm. (Mix, Huttenlocher, & Levine, 2002a*,* p. 293).

Cognitive modeling is one of the ways in which coherence and explicative power of experimental investigation can be checked and improved. Our ambition in this paper is to amount to the discussion about the nature of the AMRs system involved in quantity processing, by elaborating a conceptual space of ANS proposed by Gemel & Quinon (2015). In this way, we investigate into conceptual structure of a system of representation issued on the basis of a numerical and not spatial properties.

The model of ANS proposed by the authors is very simplified and accounts for one uniform perceptual input only (presented on a Figure 1).

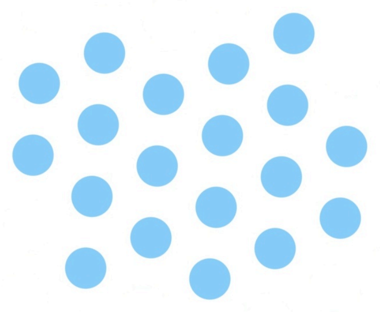


Figure 1: Uniform perceptual input for model proposed by Gemel & Quinon (2015)

Due to its simplicity, the model does not enable distinguishing reaction to the numerosity or to the space occupied by the discrete elements. In consequence, it does not provide any new input when it comes to the discussion regarding the characteristic of the AMRs system processing quantities.

The purpose of this paper is to extend the model of Gemel & Quinon (2015), by a mechanism that allows representing numerosity of non-symbolic stimuli. We extend their work by constructing a conceptual space of ANS-representations generated in response for an input of randomly distributed discrete entities of a various sizes and shapes (Figure 2).

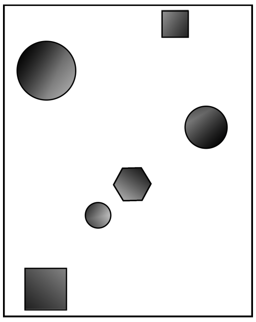


Figure 2: Non-uniform perceptual non-symbolic input

Modeling a non-uniform input necessitates an explicit account of the module responsible for processing of quantities.

2. Numerosity *versus* Magnitude

Since early 1960s researchers in the area of numerical cognition postulate existence of a priori intuitions that precede and then structure how humans, and also many non-human animals, experience and process quantitative information. Two programs are particularly well explored. Firstly, Dehaene and Brannon (2010) formulated a „Kantian Research Program" building explicitly on the Neo-Kantian philosophical framework. Secondly, Elisabeth Spelke (2000) proposed a concept of core cognition that has been largely explored for the context of numerosity by Susan Carey (2009).

There exists abundant evidence that children as young as 6 months old process quantitative information (Xu & Spelke 2000; Xu 2000). The research indicates that the system responsible for the processing most of the quantitative information is an AMRs system that is subordinated to Weber's law. That means, the ability to successfully compare two quantitative inputs does not depend on the absolute difference of sizes, but depends on the ratio. We know today that infants apprehend the difference between sets that sizes are in the ration 1:2 (e.g., 8 vs. 16 elements), but do not apprehend the difference between sets that sizes are in the ration 2:3 (e.g., 8 vs. 12 elements) (Xu & Spelke 2000). We also know that the acuity in comparison increases in individual development and adults are able to apprehend correctly ration 9:10 (Halberda & Feigenson 2008).

Systems subordinated to Weber's law are consequently subjects to two effects. According to "magnitude effect", assessment is easier for smaller items, in the case of quantities it is easier to compare 10 and 20 elements than 200 and 210. According to "distance effect" the larger ratio, the easier to correctly assess the difference, in the case of quantities it is easier to compare sets of 10 and 20 elements than sets of 10 and 11 elements. ANS is an AMRs system and our model takes into account its characteristics resulting from Weber's law.

Traditionally, ANS is considered to process numerical information:

> One central component of the number sense is the Approximate Number System (ANS). A focus of research in cognitive psychology and neuroscience, the ANS has been shown to support a primitive sense of number in infants, children, and adults (…). The ANS has been shown to produce imperfect 'noisy' estimates of numbers of items from input across all sensory modalities (e.g., beeps, visually or tactilely presented objects, taps of a finger). These numerical estimates support quantitative computations such as "greater-than, less-than," addition, subtraction, multiplication, and division. (Libertus, Feigenson, & Halberda 2011).

However, most recent research suggests that infants have a capacity of distinguishing sets of discrete elements on various bases, including surface occupied by these discrete elements, medium radius of this space, volume or density (Gebuis & Reynvoet, 2012a, & 2012b):

> In numerical cognition, the theory of the approximate number system predominates research about our ability to compare or estimate number. The theory suggests that this system enables us to extract number independently of the continuous visual variables that are present in nonsymbolic number

> stimuli (e.g., the diameter or convex hull of an array of dots) (...) As an alternative to the approximate number system, we previously proposed that nonsymbolic number judgments are based on the different visual properties comprising the number stimuli. (Gebuis, & Reynvoet, 2012a).

Results of these experiments confirm that numerosity is not the only feature of sets of discrete elements that can be used to generate representations through some AMRs system. As Leibovich et al. (2017), other researchers claim that numerosity appears at the later stage of development, small children assess quantities on the basis of spatial extension of the input. For instance, Feigensen, Carey, & Hauser (2002), observed that infants 10 and 12 month old chose the plate with crackers of a bigger entire volume even if their numerosity is smaller. In another experiment Slusser & Sarnecka (2011) tested 116 infants of 30 to 48 months, on a task consisting on showing the infants a n image of a herd of big turtles and asked to decide if this picture matches with an image of the same number of smaller turtles or rather with an image of twice as numerous herd of small turtles that were covering the same surface as the big picture from the initial image. Only children having a certain experience with counting and understanding names of numbers could correctly perform this task.

The theory of Mix et. al. (2002b) is in line with this view. Mix's thesis is based on the assumption that infants discriminate and quantify sets by using perceptual non-numerical cues such as the volume, area, volume, length. In consequence they reject a double nature of ANS and claim that all processing of non-symbolic quantitative information takes as input continuous values. The authors write:

> development starts with only one principle of quantification in infancy based on amount of substance, which applies to both continuous quantity and sets of discrete objects (Mix et. al. 2002b, 62),

and then they continue:

> infants do not represent quantities numerically at all. Instead, the evidence points to the use of overall amount. (Mix et. al. 2002b, 81).

Observe, that the discussion concerns a nature of representations generated by ANS. Proponents of both theories agree that the input contains both discrete (numerical) and continuous information. It is also commonly assumed that there exists an AMRs system that transforms exact information of the input into an approximate representation. What is unclear is which information is represented as an output.

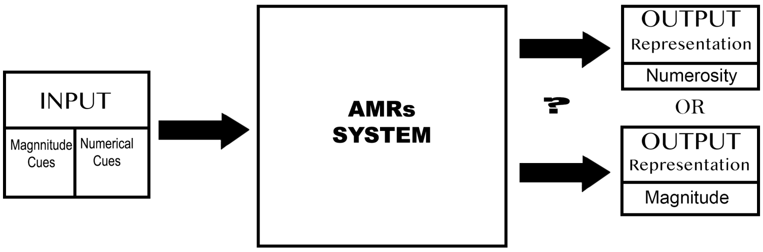


Figure 3: Scheme of possible representational structure of the AMRs system

Numerical reading is postulated by Dehaene (1997/2011) who claims that these representations are generated from the discrete input by ANS. Hence, ANS evolved to detect numerosity of the apprehended input and to abstract from its magnitude related cues, such as size of compared objects, their cumulative and overall area, and density of objects location. For instance, when we see a pile of apples, ANS acts as a "number sense" that informs us about the approximate quantity of apples. According to & Ross (2008) and Nieder & Dehaene (2009) this process is as basic and automatic as perceiving the perceptual features of objects. In other words:

> Perceived numerosity is susceptible to adaptation, like primary visual properties of a scene, such as color, contrast, size, and speed. Apparent numerosity was decreased by adaptation to large numbers of dots and increased by adaptation to small numbers, the effect depending entirely on the numerosity of the adaptor, not on contrast, size, orientation, or pixel density, and occurring with very low adaptor contrasts. We suggest that the visual system has the capacity to estimate numerosity and that it is an independent primary visual property, not reducible to others like spatial frequency or density of texture. (Burr & Ross 2008, 425).

Prevalence of the numerical aspect of representation is particularly clearly visible in the computational model proposed by Dehaene and Changeux (1993). Input of this model is a non-symbolic, possibly non-uniform, visual stimuli (such as a group of objects shown on Figure 4), and the final output is an approximate representation of the numerosity of elements in the input. The model of Dehaene and Changeux (1993) is based on the connectionist approach, which implies modeling by artificial neural networks. The modeling process runs through several phases. In the first phase, objects are encoded according to their location in space, corresponding to information from retina. It is modeled by the topographic map of object locations. In the artificial neural network it is represented by activation of neurons. At this phase abstraction is made from all the features related to continuous magnitudes (like the size of items, density of the set, *etc.*). In the second phase, model estimates the numerosity of items represented in the first phase thanks to numerosity detectors composed of groups of neurons specialized in detecting information on the number of items in the stimuli (Dehaene & Changeux, 1993, 394). This reflects the assumption that capacity of detecting the numerosity is innate. The role of numerosity detectors is to sum up the value of neuronal activation of active neuronal groups in the previous phase. In the final phase the neural network generates approximate representation of inputs numerosity. Model was tested and supported in the work of Verguts and Fias (2004), who showed that after training the network, the distribution of errors in both discrimination and the comparison tasks followed experimental data regarding two ubiquitous effects in numerical cognition (*i.e.* size and distance effects).

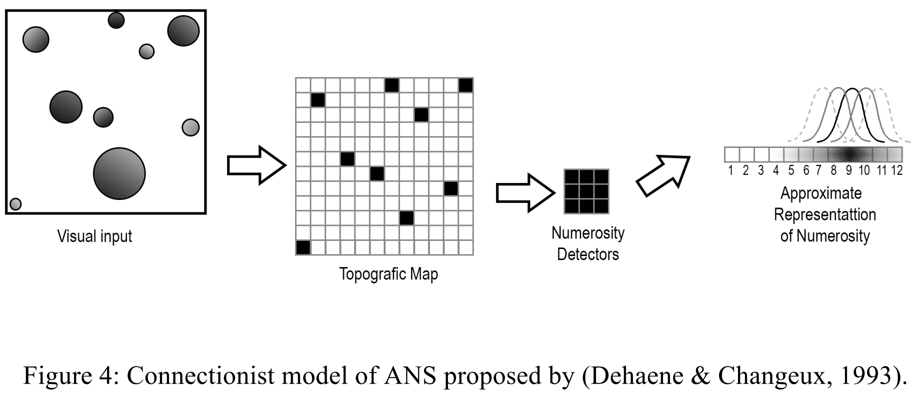


Figure 4: Connectionist model of ANS proposed by (Dehaene & Changeux, 1993).

According to the defenders of “continuous magnitudes” reading, there exists no system, as ANS under its classical interpretation that processes numerical information and generates continues representations encoding information about the quantity of discrete elements in a collection. “Number sense”, if any, develops in ontogenesis from systems processing continuous magnitudes. In its place there exists an innate “sense of magnitude” enabling humans (and also many animals) to differentiate between continuous magnitudes, *i.e.*, the whole surfaces that the assessed items cover (Gebuis & Reynvoet 2012b; Leibovich & Ansari 2016; Leibovich et al. 2015; Mix et al. 2002).

In theories exploring the “sense of magnitude”, the processing of non-symbolic quantitative input is holistic, *i.e.,* numerosity in not extracted from non-symbolic input independently from the information about the surface magnitude (size of objects in a stimuli, cumulative and overall area of a stimuli and density of stimuli). All these types of information are processed simultaneously and in consequence the generated representations have a double nature. As claims Leibovich, this is how actually humans process information:

> We argue here that since it is physically impossible to study discrete magnitudes in isolation, it is difficult to accept theories such as the ANS and the number senses, which suggest numerosity is the most salient cue. It is more likely that we integrate multiple visual cues to make size estimations (…). Following this logic, it is more instrumental to use all available cues in the environment – not only discrete ones – to make size estimations. (Leibovich, & Henik, 2013, p. 2).

Based on the idea, that there is a natural correlation between continuous magnitudes and discrete magnitudes (numerosity), and that all types of information are used holistically to judge numerosity Leibovich, & Henik proposed a model of AMRs system in a developmental framework. Their model is intended to describe how magnitudes are represented and discriminated at different developmental stages (Figure 5):

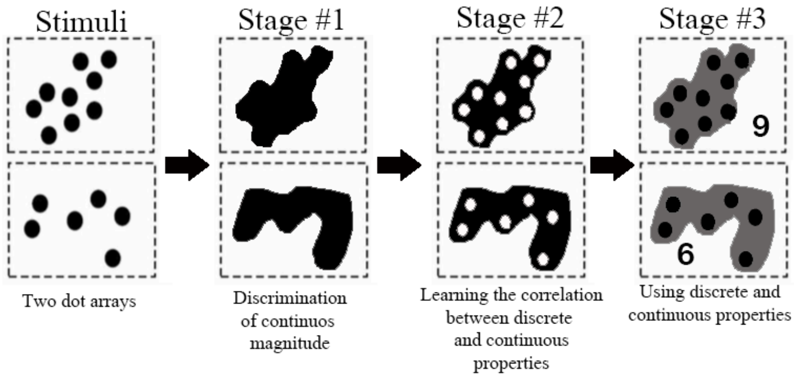
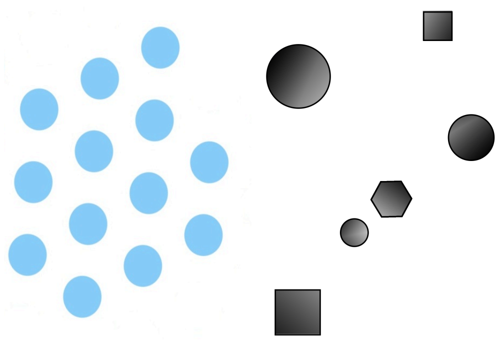


Figure 5: Developmental model propose by Leibovich, & Henik (2013).

According to Leibovich & Henik's model, humans are born only with the ability to discriminate between continuous properties. At the first stage of development, infants judge the stimuli relaying on continues magnitude only, because their vision is not acute enough to focus on specific items. During the development - after infants reach the age of 5 months and they will be finally able to individuate items from the background and from one another - babies represent discrete and continues properties, and they start to learn the correlation between both properties. After that babies are able to use both discrete and continuous properties to estimate magnitudes. On the last stage of development, when they start formal mathematical education, children are able to represent the exact difference between different magnitudes. However even after formal training the representations of non-symbolic stimuli processed by the AMRs system is holistic (*i.e.,* takes into account both, numerical and magnitude aspect of the input). In other words, the output of their model are representations that convey information regarding the discrete quantity of elements in the assessed collection and the surface they cover.

None of the readings – discrete and continuous magnitude – gathered sufficient empirical support that would fully exclude the other. In consequence, each theoretical model – connectionist, developmental and also the one in conceptual spaces – necessities a decision which type of input – discrete or continues – prevails. A possibility of formulating a consistent model might help experimental research to formulate hypothesis in one or another direction. In the model that we propose, representations are issued from discrete input.

We improve the results from (Gemel & Quinon 2015) by showing how the number related cues can be represented in a model of ANS in conceptual spaces. According to the authors, representations generated by the model contain purely quantitative information, but from their presentation there is no way to distinguish numerosity related cues from magnitude related cues. Differentiation of the two types of information is not possible because the model is defined for one type of uniform perceptual input only (presented on Figure 6(a)) where the number of elements and the surface occupied by them change proportionally. In this paper, we propose to extend the model and propose a conceptual space that will depict conceptual structure of numerical information generated by ANS. We will work with a non-symbolic, non-uniform input (Figure 6(b)).



(a) (b)

Figure 6: (a) Uniform perceptual non-symbolic input, (b) non-uniform perceptual non-symbolic input

Our objective consists in showing a coherent conceptual picture generated by ANS processing quantitative information. Although, we will let aside general questions relating cognitive processing of quantitative information and we will not provide any definitive answer which theory is correct and how number and magnitude sense are correlated, conclusions of our endeavor will amount to a better understanding of the structure of those processes.

In particular, we extend model proposed in Gemel & Quinon (2015) by explicitly defining a number-specific module sensitive to numerosity of the stimuli. This number-specific module comes is inspired by Dehaene & Changeux (1993), who postulate the existence of a similar mechanism, which they call "numerosity detectors". Model by conceptual spaces will enable us to avoid some major problems that encounter connectionist model of Dehaene & Changeux. First of all, it will avoid the ubiquitous drawback of every model based on an artificial neuron network's architecture consisting of a necessity of providing a large training set to develop the relevant structures, and to learn to represent an information. The reason why it learns so slowly is that it consist a large number of learning parameters. In other words all the weights of neuronal connections are treated as independent variables, and all of them have to be adjusted during trained of the model (Gärdenfors, 2000). This is mainly the reason Dehaene & Changeux' model is limited only to 5 elements (Dehaene & Changeux 1993, p. 396).

Secondly, it is very hard to express a conceptual change in connectionistic models, since the reconfiguration of the network is quite difficult and time-consuming process. Consequently the model becomes hard to use in any developmental framework, for instance in a theory of accurate numerals development.

Third drawback is a fundamental epistemological problem of artificial neuronal modeling. ANS is basically only a tool of emerging representations. It actually does not give us the knowledge of representational content, but the knowledge on how the representation is implemented in neuronal structure of the brain. In other words even if we are aware that a network categorize the input in an appropriate way, we may not be able to describe what it actually represents.

These difficulties may be avoided by constructing the model at a higher, conceptual, level of representation. Conceptual space is the theoretical framework for such level of representation. Therefore the aim of the next chapters is to present conceptual space model for ANS representational structure.

3. Number-Sensitive Module in Conceptual Spaces

The model proposed by Gemel & Quinon (2015) could not deal with non-uniform input, where elements are not all of similar size and are not distributed in a regular manner. In this paper, we extend this model of an additional number sensitivity module – inspired by Dehaene & Chagneux (1993) “numerosity detectors” – that transforms discrete non-uniform input into a continuous representation of numerosity.

Our model consists of three phases. In the first one is made the abstraction of magnitude related cues of the input stimuli. Second phase encodes numerical information from first phase to continuous, analogue representation. Finally, the last phase aligns the outputs of the second phase according to the prototype structure of the category.

3.1 First Phase: Topological Representation

In the first phase, visual input (Figure 1) is represented by a group of points in two-dimensional similarity space encoding the spatial location of objects contained in the stimulus. The similarity space in which the stimuli is represented is defined by two quality dimensions: height and length. On this space the similarity relation is represented by a city-block metric defined by the following equation:

$$d_{Cb}(x,y) = \sum_{k=1}^{n} | x_k - y_k |$$

On this metric $d_{Cb}(x,y)$ signifies distance between two points x and y. Variables x_k and y_k represent the coordinates of the points x and y on the quality dimension axes (k stands for quality dimensions of the similarity space, which in our case are height and length, n stands for the number of the quality dimension constituting the similarity space, which in our model is two). In less formal terms, in a city-block metrics the measure of the distance between two points is the sum of absolute values of arithmetical differences between their coordinates. The metric provides us the information about how many degrees in the height dimension and how many degrees in the width dimension points are spaced apart. Degrees on the scale in the proposed metric are defined by the relations of "being-above" and "being-below" for the height dimension, and by the relations of "being-to- the-left" and "being-to-the-right" for the length dimension. Scale defined by these relations is intended to represent the distance between two objects measured by the number of elements between them. To put it simply, if two points (x

and y) are separated by one degree on a scale on the vertical axis, then y is directly above the x and x is directly under y.

The basic relations describing scales in such a conceptual space are defined as:

Above:

(Ax 1) $\forall_x \forall_y \, xAy \Leftrightarrow \neg yAx$

(Ax.2) $\forall_x xAx \Rightarrow x \in \emptyset$

(Ax.3) $\forall_x \forall_y \forall_z xAy \wedge yAz \Rightarrow xAz$

The *Beyond* relation is dual to relation of being *Above.* The relationship of being *Above* is gradual, so we can introduce the relations of being *Above on one degree*, i.e. (A_{1°), which we can define as:

$$\forall_x \forall_y \; xA_{1^\circ}y \Leftrightarrow xAy \wedge \neg \exists_z \, zAy \wedge zBx \, ,$$

In the same way, we can define a relation of "being-on-the-left" (L_{1°) and "being-on-the-right" (R_{1°).

Relations A_{1° and B_{1° define the degrees of similarity on the height dimention, and L_{1° and R_{1° relations on the lengh dimention in our similarity space.

The scale in such defined space is thus a normalized measure of the distance between two points x and y, since it takes into account only the number of elements distant from x, and not the overall physical size of distance.

Representation of objects from the input in this conceptual space allows to abstract from most of the magnitude related cues of their representation (*i.e.* their size, their shape, overall area occupied by them, etc.). Each object is in fact represented by a point in space. Moreover, city-block metric allows us to abstract from information about physical distances of objects contained in the stimulus. The first phase of the modeling of stimuli representation in the conceptual space results in a structural map of the direct neighborhood relations between objects in the stimuli. Relation of direct neighborhood can be defined as:

$$\forall_x \forall_y \; xDNy \Leftrightarrow xA_{1^\circ}y \vee xB_{1^\circ}y \vee xL_{1^\circ}y \vee xR_{1^\circ}y$$

First phase of modeling is presented at Figure 7. Note that, the topological structure of representations preserves topological structure of representations of the stimuli.

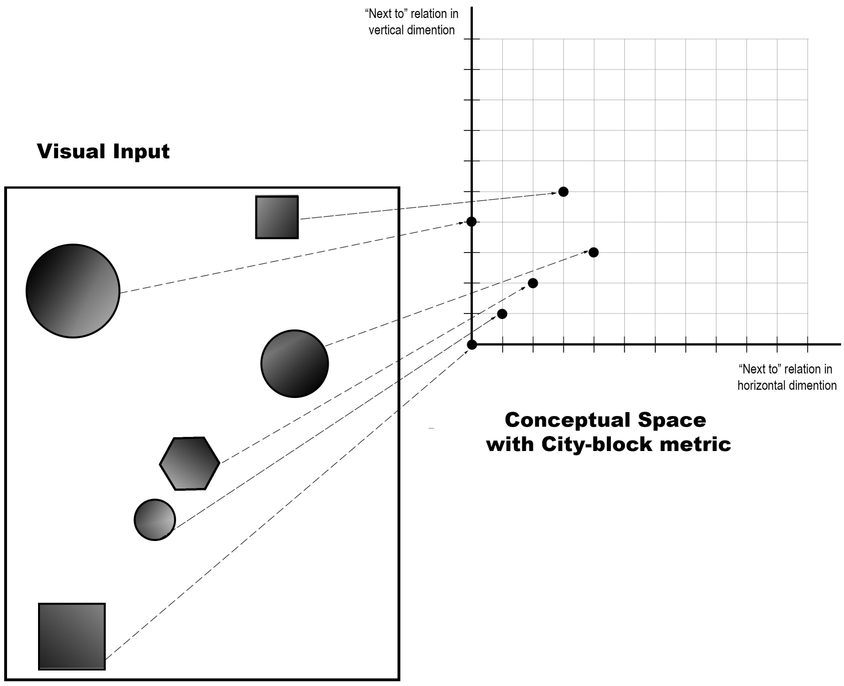


Figure 7: First phase of heterogenic stimuli representation in conceptual Space

3.2 Second Phase: Analog Magnitude Representation of Numerosity

The second phase of modeling of the representations generated by ANS consists in encoding how numerical information gets translated into continuous, analogue representations. One simple way of thinking about this kind of representation is proposed by Carey and Sarnecka (2006, p.477): "[there is] a helpful analogy to the following external system of analogue number representations. [...] Line length is a direct analogue of number" (Figure 8).

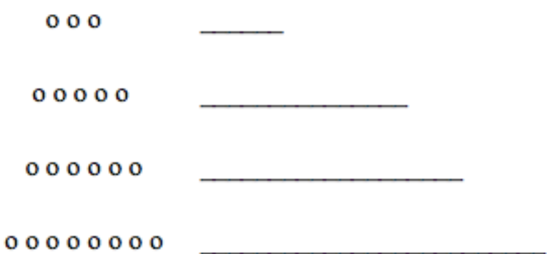


Figure 8: Analog representation of numerosity of non-symbolic stimulus

In other words, term “analog” can be understood in accordance with Goodman’s (1976) view, *i.e.* in terms of being continuous, or being *dense*. This means that between any two represented values there is always a third represented value (*Ibidem*). Therefore, best tools for a description of these representations seem to be spatial categories and hence spatial quality dimensions. The size of the representation (*i.e.* line length) is proportional to the quantity of the elements in the collection (numerosity of a stimuli). The bigger the quantity of the input, the bigger the magnitude represented in the system. In the second phase, representations generated in the first phase are transformed in analogue magnitude representations.

Thus, the second phase consists in the representation of the stimulus from the first phase (i.e. the normalized set of points) as a polygonal chain running through each of the points of that set. This phase is carried out in two steps. First the topological representation of the stimulus is scaled to the Euclidean metric defined on the similarity space as:

$$d_E\,(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

In the Euclidean Metric, the distance between any two points x and y is the length of the segment connecting them: $\overline{xy}$.

Next, we assume that a simple algorithm running on the similarity spaces can generate polygonal chain. The line can be formed by a clockwise ordering of h points L_0 ... L_{h-1}. The first point L_0 is typically the leftmost point in the set P, if multiple points exist in P with the same x coordinate, then L_0 is the one with the smallest y coordinate, although any point can be the start. The algorithm starts at step $i = 0$ and determines the first point of the chain L_0, with the co-ordinates of the lowest values on both axes. The next point L_{i+1} is the one with the smallest distance to L_0. Measurement of the distance between points is carried out in accordance with the Euclidean metric. It gives us guarantee that the shortest route will connect the consecutive points. Moreover, as it is shown on figure 9, in Euclidean metric, unlike as in City-block metric, there is only one possibility of connecting two points laying diagonally. Thus Euclidean metric gives us the guarantee that the merging of consecutive points will be done in an unambiguous manner.

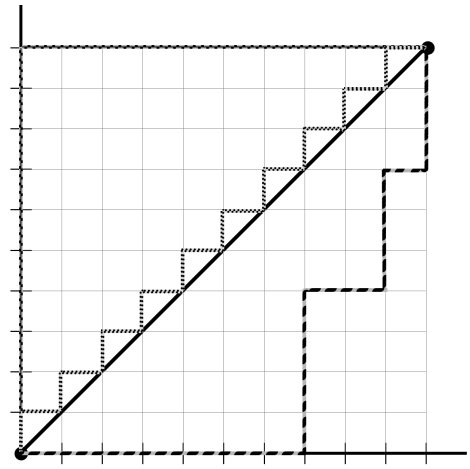


Figure 9: Same distance in Euclidean metric (black solid line), and several equivalent distances in City-block metric (gray dotted lines)

In case when more than one point is adjacent to L_0 (for instance, when L_0 has (1,1) coordinates and coordinates of points L_i and L_j are (1,2) and L_j (2,1) respectively), then L_{i+1} is the point with a lowest value on the ordinate axis. Following steps of the algorithm are based on an iterative recursive procedure for the step defined as $i: = i+1$, executed in number of h steps repeated until the end point is reached, *i.e.*, when $L_h: = L_h + 1 = L_0$. The last step of the algorithm is thus the one preceding the return to the starting point (L_0).

Of course, the efficiency and processing time of the algorithm should be checked experimentally. However, it seems that due to the normalized layout of the points generated by the output of the topographic map module its computational complexity will be small enough to fully agree with human computational resources and with rules of cognitive economy.

The second phase of the model transforms a collection of discrete points from the first phase into continues representation, compatible with analog magnitude code, as depicted at Figure 10.

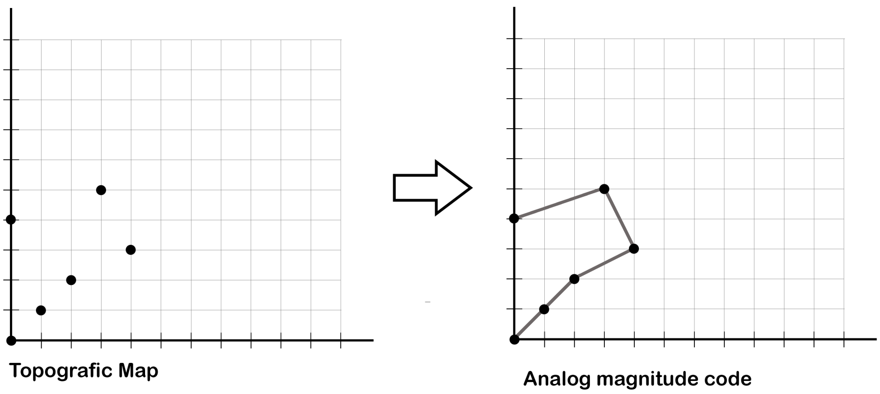


Figure 10: Transformation of topographic map into continues representation, compatible with analog magnitude code

It is worth noting that the numerosity of a given representation (*i.e.*, the number of elements on which the algorithm operates) is directly proportional to the number of steps that algorithm executes to generate the polygonal chain on all the points (*i.e.*, chains vertexes). In consequence, even if continuous representations generated by our model might significantly differ even for equinumerous input (Figure 11 depicts that situation for several representations of the stimuli composed of 6 elements) each of the continuous representations will eventually have the same numerical dimension (corresponding to the number of steps in an algorithm).

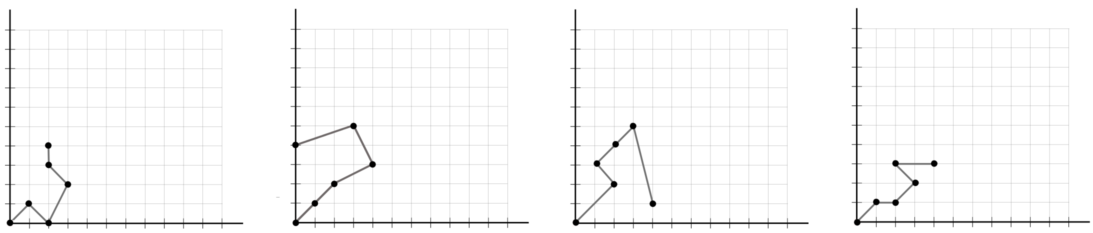


Figure 11: Possible AMRs of equinumerous input.

The numerosity dimension arises from the information how many steps the algorithm needs to make in order to connect all the dots. Numerosity qualitative dimension is therefore represented by another axis *Z* (*applicata)* next to the ordinate *Y* and abscissa *X*. This axis is isomorphic to the half-line of positive numbers (*i.e.* without negative numbers and zero), since there is no possibility to represent negative numerosity and zero through non-symbolic stimuli. Moreover, numerosity dimension represents only numerosity of the stimuli when it consist of more then one element, since there is no possibility of an approximate representation of individuality by Approximate Number System (Feigenson, *et. al.* 2004; Xu, 2003). This is the reason why initial point of the half-line is one instead zero.

The steps that the polygonal chain generation algorithm performs define degrees on a scale of the axis representing the numerical dimension. This means that the numerical value of the stimulus is always one degree greater than the number of steps taken. One degree on the scale of a numerical dimension means that the algorithm has taken one step to connect two points, so numerical value of a stimulus equals two. The numerical dimension is presented on Figure 12.

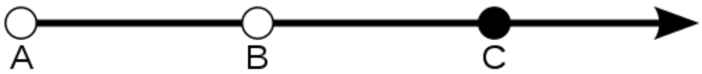


Figure 12: Numerical dimension – A represents 0 steps of polygonal chain algorithm which correspond to numerosity of stimuli consist one or zero elements; B represents 1 step of algorithm, which corresponds to numerosity of stimuli consist 2 elements, *etc.*

Naturally, representations of numerosity on this quality dimension are encoded in analog form. For instance, one step in the polygonal chain algorithm is represented as a continuous line of a one-degree long on the numerical dimension. Since the scale on numerical dimension starts with one, the line will have a length of two degrees. This value corresponds with numerosity of the stimuli.

We assume that, in the analogy to the solution proposed by Dehaene & Changeux, numerical representation is generated as a result of summation. In our proposal, the algorithm generating polygonal chain sums up number of its own steps, and represents them as a line in another quality dimension. This dimension represents the numerosity as a length of a line corresponding to the numerical value of the stimulus. The numerical value of a given *n*-element stimulus will always be represented as a line of length equal to $i = n-1$, since the number of steps i is always 1 less than the size of the stimulus n.

The model of analogue magnitude representation of numerosity generated in the second phase is a three-dimensional similarity space consisting of dimensions of length, height, and numerical dimension. See Figure 13:

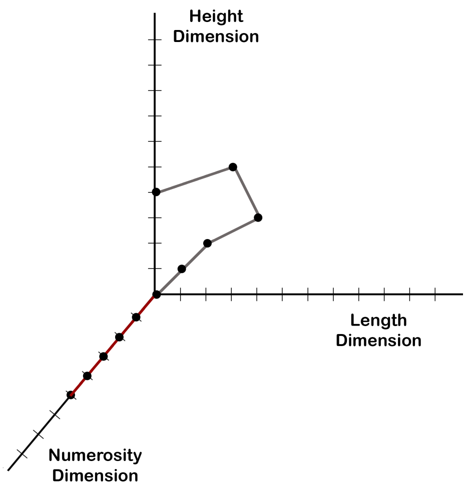


Figure 13: The output of the Second phase of modeling - analogue magnitude representation of numerosity

However, it should be noted, that while different stimuli may have the same numerical value, some of them would be more easily recognizable and distinguishable from others. This means that the cognitive effort needed to distinguish the numerosity of two stimuli depends not only on the number of objects, but also on their arrangements. For instance, compering the numerical value of two stimuli with linear arrangements (*Stimuli 1* on Figure 14) of objects will be easier, (*i.e.* will involve less cognitive effort) than distinguishing the stimulus with linear arrangements of objects from the stimulus in which objects are arranged randomly (*Stimuli 2* on Figure 14).

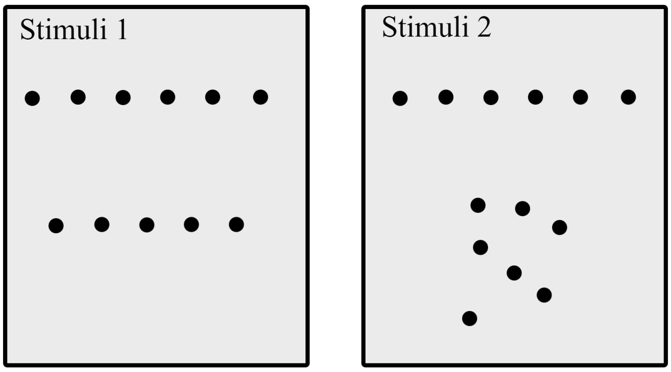


Figure 14: Differences in the difficulty of distinguishing different objects' arrangements in the stimuli.

Our model provides an intuitive way to explain this phenomenon. It provides a mechanism for ordering ANS representations, according to the degree of complexity of objects' arrangements in the stimulus. This mechanism is based on the prototype structure of the representation.

According to prototype theory, the internal structure of a category is not homogeneous, which means that there may be instances of different degree of representativeness within it. Some elements are therefore more centrally located within categories than others. These centrally located elements, called prototypes, constitute a specific measure of categorical affiliation. Elements more similar to the prototype are situated closer to the center of the category, and characterized by a greater degree of membership, while those less similar to the prototype take a more peripheral location in the internal structure of the category. A mechanism for organizing the ANS representations in accordance with the prototype organization of category is described in the next paragraph.

3.3 Third phase: Prototypical representations of ANS representations.

The third phase consists in organizing representations depending on their similarity to the prototypes. We consider that certain types of distribution of the kind of visual input modeled in our study are cognitively privileged. Similarly to what was proposed by Gemel & Quinon (2015), we justify existence of prototypical cases of quantitative representations on the basis of psychological investigations.

The first phase of our model preserves topological structure of representations of the stimuli (presented at Figure 7). This topological structure generated in the first phase of our modeling (see Figure 7) is also preserved by the second phase of modeling (see Figure 13) where it also got enriched by additional quality dimension of numerosity, encoding the numerosity of the stimuli. Note that each such representation is associated with an analog representation of the topographic structure of the arrangement of objects in a given stimulus, which is represented by a polygonal chain in a two-dimensional space consisting of height and length quality dimensions. While different configurations of the stimuli with the same numerical value do not differ in terms of size on the numerical dimension, their topographic representations are completely different (Figure 11). These analog topographical representations have direct impact on a variation of cognitive effort needed to estimate the numeracy of a given collection.

We use topographic properties of the stimuli to define the prototype structure of ANS-generated representations. Prototype structure reflects the fact that an estimation of numerosity of the prototype will be much easier than a peripheral element. Moreover, it is more likely that the latter will be estimated incorrectly. Some arrangements of elements in the stimuli represented by the polygonal chain will be more easily – in terms of cognitive effort reflected by a time needed to fulfill categorization task – recognized by the subjects as corresponding to a given numerosity.

Obviously, empirical research should be conducted to determine the actual prototypical arrangements. Gemel & Quinon (2015) propose that prototypical arrangement is the most evenly distributed one, *i.e.* a square-like arrangement of equal height and length dimension ratio. They justify their proposal on the basis of Tversky's (1977) studies on prominence of geometric figures. According to Tversky more prominent or salient figure, the more regular and symmetric shape it has. However, Gemel & Quinon proposition was not based on empirical research.

Before an empirical study is conducted and a set of distribution recognized as prototypes psychologically grounded, we do not want to prejudge the actual shape of prototype. For simplicity, we propose that in our model the role of the prototype is played by straight-line arrangement of objects. However at this stage of research we do not exclude the hypothesis, that dots distributed in different way are in fact more prototypical, or that more than one shape will be prototype. Obviously these cases will significantly affect similarity relationships within the conceptual space, and consequently also on the final shape of the representation structure generated by the ANS. In this paper, we propose a theoretical and hypothetical representational structure of the prototypical structure of the ANS generated representations. Hypotheses presented in this paper could be tested empirically in series of categorization experiments. One of our ambitions is to prepare theoretical and conceptual frameworks for such studies. We are aware of limitations that this constraint entails and of the necessity of adjusting the proposed model to empirical results on prototypically. However proposed model is easily adaptable to a model consistent with an empirical data.

The third phase of our model consist in ordering a collection of all possible configurations of analogue continues representations associated with a given numerosity (for instance 6) according to similarity to prototypical elements of the collection. Recall, that output of the second phase of modeling is continues representation in three-dimensional similarity space consist of two spatial quality dimensions (height and length) and one numerical. The ordering of the representations proposed in our model is made in purely spatial terms, *i.e.* through degrees of similarity in height and width dimensions of the polygonal chains generated by the second phase of the model. The set of prototypes consists of all possible linear arrangements of objects in the stimuli (*i.e.* horizontal, vertical, and diagonal lines). Figure 15 depicts such an ordering for collections of 6 elements.

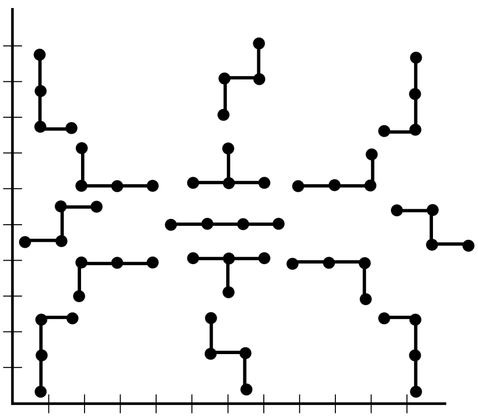


Figure 15. Ordered set of some possible arrangements of 4-element collections of ANS' stimuli.

By introducing a prototype structure we can create a entire structure of all representations generated by ANS. Structures of representations generated by ANS for other quantities are modeled in the same way. The model of entire structure of conceptual representations generated by ANS is achieved by positioning models of individual quantities in one common space. It can be shown that such a common space for all ANS' representations of quantities can be obtained by a Voronoi tessellation.

A Voronoi tessellation is a modeling technique proposed by Gärdenfors to carve up the prototypical structure of given conceptual space. Formally it can by express by formula

$$VorP(p_1)= \{x \in E | \forall p_n \in S, d(x,p_1) \leq d(x,p_n)\}$$

where $S=\{p1,...,pn\}$ is the set of prototypes, (generators), E, stands for Euclidean metric space, and d is a distance function on E correlated with relation of similarity between stimuli. In other words, if we start from a set S of prototypes represented by the points (p_1, ..., p_n) and if we assume that E is a Euclidean metric space, than for every point x in the space we can actually measure the distance from x to every element of the set of prototypes. Partitioning of the space is than generated by the rule that: x belongs to the same region P as the closest prototype from set S.

However we assume that in our model there will be at least several prototypes assigned to a given numerosity. If we stipulate, that linear shape is prototypical, than the possible linear arrangements of elements in the stimulus are horizontal, vertical and diagonal. At this stage of research we also do not want to exclude that shapes similar to line (like linear arrangement of five points and one point above or below them) will also be prototypical. Moreover we assume that there will be a growing number of prototypes for representations of growing numbers.

Since in our model the number of prototypes assigned to a given number is greater than one, the division of similarity space is not made by the set of prototypes represented by the points, but by the set of subset of prototypes represented as set of collections of points. In short, this model requires using as set of generators of subsequent tessellations a set of clusters of points $\{\{p1_1, \ldots, p1_n\}, \{p2_1, \ldots, p2_n\}, \ldots, \{pn_1, \ldots, pn_n$

}} instead of a set of points $\{p1_1, \ldots, p1_n\}$. As it was shown by Douven *et al.*, (2013), this type of tessellation generates blurred borders between the representations. Prototype groups model proposed by Douven et al. enables modeling of the approximate nature of ANS generated representations. As a result, we get the following diagram:

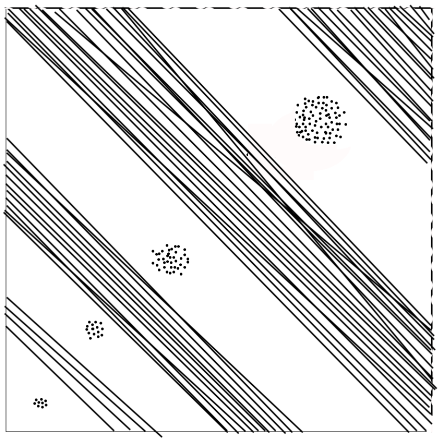


Figure 16: Structure of ANS representations for successive numerosities.

Note, that this picture is simplified visualization of ANS representational structure. We assume that the boundaries between the successive representations will become wider as the progressions of numerosity increases. We also assume that for large numerosities the boundaries between representations can overlap more than one area. This outcome is consistent with logarithmic distribution of the compressed numerical line of ANS representations (Figure 17):

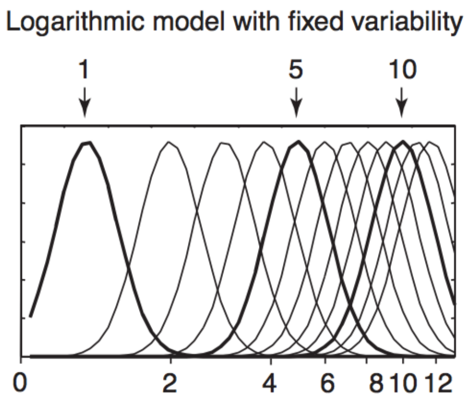


Figure 17: Logarithmic distribution of ANS representations (Feigenson, Dehaene, Spelke, 2004)

4. Conclusions

The model of ANS in conceptual spaces proposed in this paper reflects the specifics of the modeled system and it accounts for size and distance effects, characteristic for ANS and are in accordance with Weber-Fechner law. Furthermore, proposed model is sensitive for number related cues, and can be easily reconciled with the sub-symbolic computational model of Dehaene & Changeux, while avoiding numerous problems of the sub-symbolic level of representation. We believe that further investigation into the exact structure of this model is necessary, in particular taking into account experimental data disclosing sets of empirical inputs, which correspond to prototypical representations.

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