Conceptual Content of Arithmetical Terminology

From Number Cognition to Mature Mathematics

Paula Quinon

Numbers in Mind: The Philosophy of Numerical Cognition
University of London, Feb 18, 2017

Outline

1. Ways of using cognitive science in philosophy
2. Framing questions in two-folded picture: syntax and semantics
3. Conceptual content: syntactical and semantical
4. Conceptual gap: how to account for regularity?
5. Enculturation and extended mind
6. Extended Frege's Constraint and Optimality Principle

Use of cognitive sciences in philosophy

- Folk psychology to support one's epistemic position (Wright, Shapiro, Maddy)

- Embodiment (Nunez and Lakoff)

- Philosophy of mathematical practice

- Experimental philosophy

- Metaphysical investigations (Panza, De Cruz)

Conceptual engineering

Conceptual engineering is a powerful multi-tool method aiming at sharpening, reviewing and/or improving concepts, which keeps ideals of conceptual analysis, but is more inclusive and comprise other methods such as the method of explication, the method of scientific explanation or the theory of conceptual change.

>> Cappelen, Linnebo, Eklund

Frege's Constraint

Foundations (axiomatic system or set of first principles) of a mathematical theory are satisfactory when they account for applications of the entities forming the intended model of this theory

Candidate constraints:
- cardinality
- computability
- omega-structure

Applications of natural numbers

- **Cardinality:** natural numbers serve to account for cardinalities of finite sets (Hume's Principle)

- **Computability:** natural numbers are byproducts of computation processes (Tennenbaum's Theorem)

Carnapian Explications

"we mean the transformation of an inexact, prescientific concept, the *explicandum*, into a new exact concept, the *explicatum"* [page 3]

Carnap 1950

Second philosophy

"This Second Philosopher is equally at home in anthropology, astronomy, biology, botany, chemistry, linguistics, neuroscience, physics, physiology, psychology, sociology,...and even mathematics, once she realizes how central it is to her ongoing effort to understand the world. Her interest in other subjects, at least as far as we see her here, is limited to her pursuit of their anthropology, psychology, sociology, and so on." [page 2]

"… she begins from common sense, she trusts her perceptions, subject to correction, but her curiosity pushes her beyond these to careful and precise observation, to deliberate experimentation, to the formulation and stringent testing of hypotheses, to devising ever more comprehensive theories, all in the interest of learning more about what the world is like. She rejects authority and tradition as evidence, she works to minimize prejudices and subjective factors that might skew her investigations." [page 14]

Maddy 2007

Enculturation

"Enculturation rests in the acquisition of cultural practices that are cognitive in nature.The practices transform our existing biological capacities, al low ing us to complete cognitive tasks, in w ays that our unenculturated brains and bodies will not allow." [page 4]

Menary 2015

“Humans possess two nonverbal systems capable of representing numbers, both limited in their representational power: the first one represents numbers in an approximate fashion, and the second one conveys information about small numbers only.“

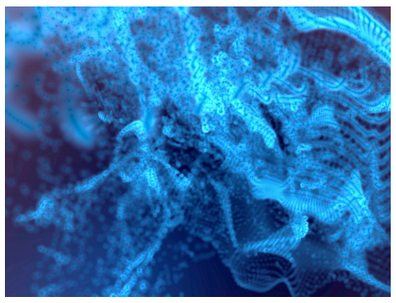
Izard et al. 2008, 491

Humans possess two nonverbal systems which enable humans to react appropriately to quantitative information and provide conceptual content for numerical expressions, both limited in their representational power: the first one provides content of quantity in an approximate fashion, and the second one conveys information about small quantities only.

Syntax and semantics

Syntax and semantics

Concrete objects
Strings of characters
Inscriptions-Numerals-Notations



Abstract objects
Natural numbers
Numbers

→
Denotation function
δ

String-theoretic functions
$S \rightarrow S$

$S \rightarrow \mathbb{N}$

Number-theoretic functions
$\mathbb{N} \rightarrow \mathbb{N}$

Turing's thesis
a string-theoretic functions
is computable
iff it is Turing-computable

Church's thesis
a number-theoretic function
is computable
iff it is recursive

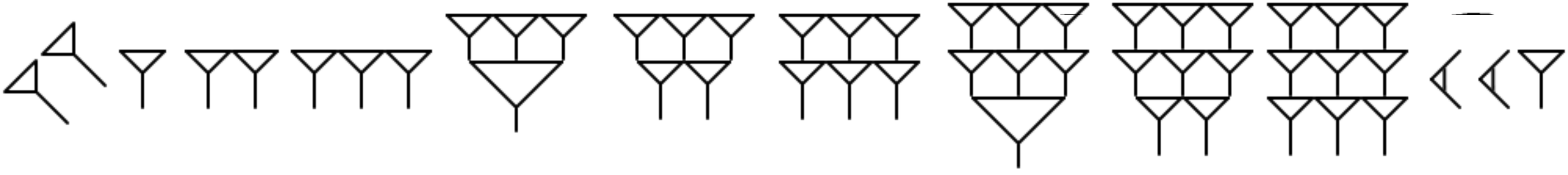
Syntax

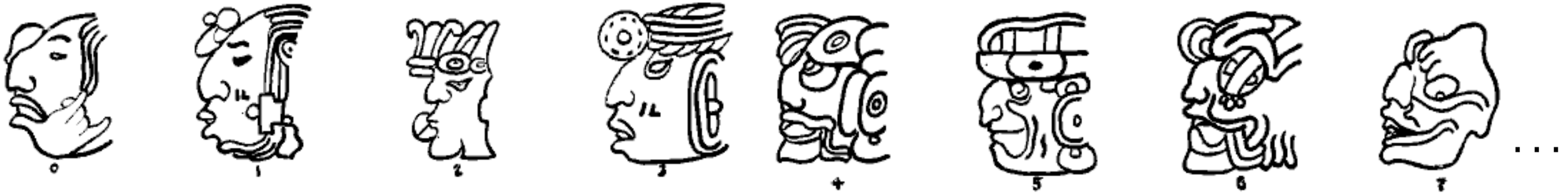
Part 1

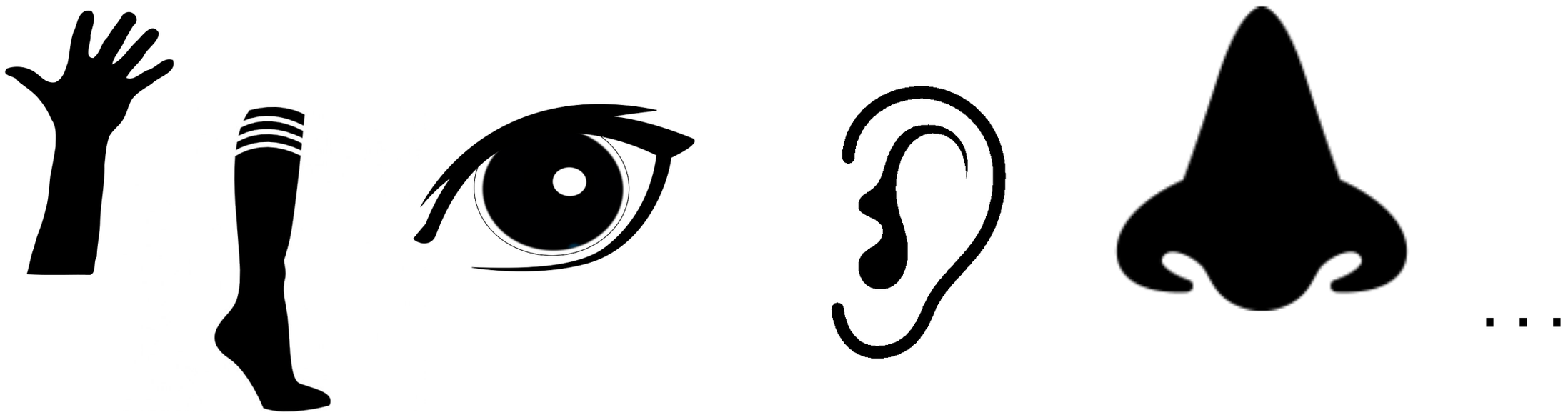
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, …

e, I, II, III, IIII, IIIII, IIIIII, …

∅, {∅}, {∅,{∅}}, {∅, {∅}, {∅,{∅}}}, …







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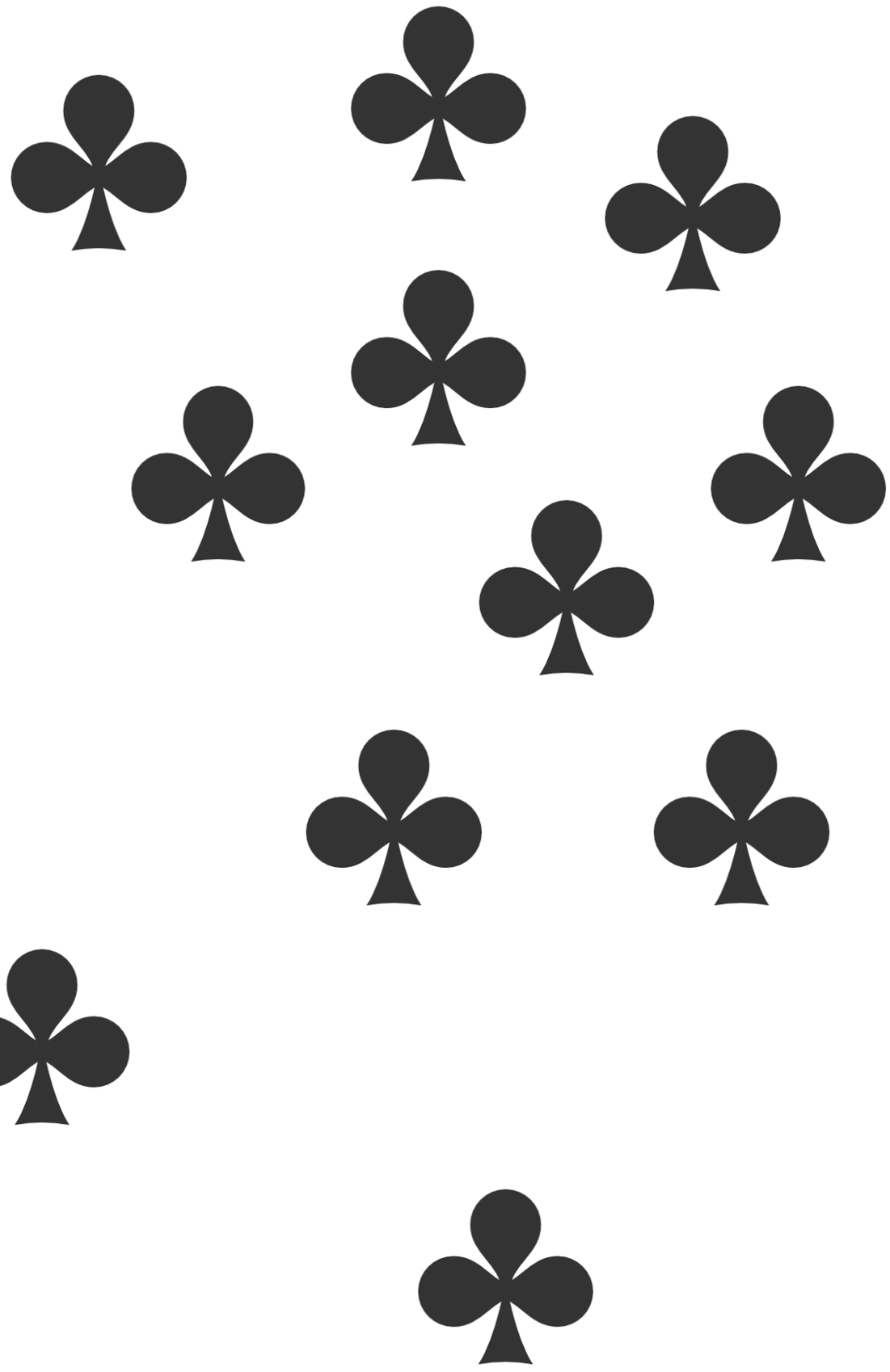
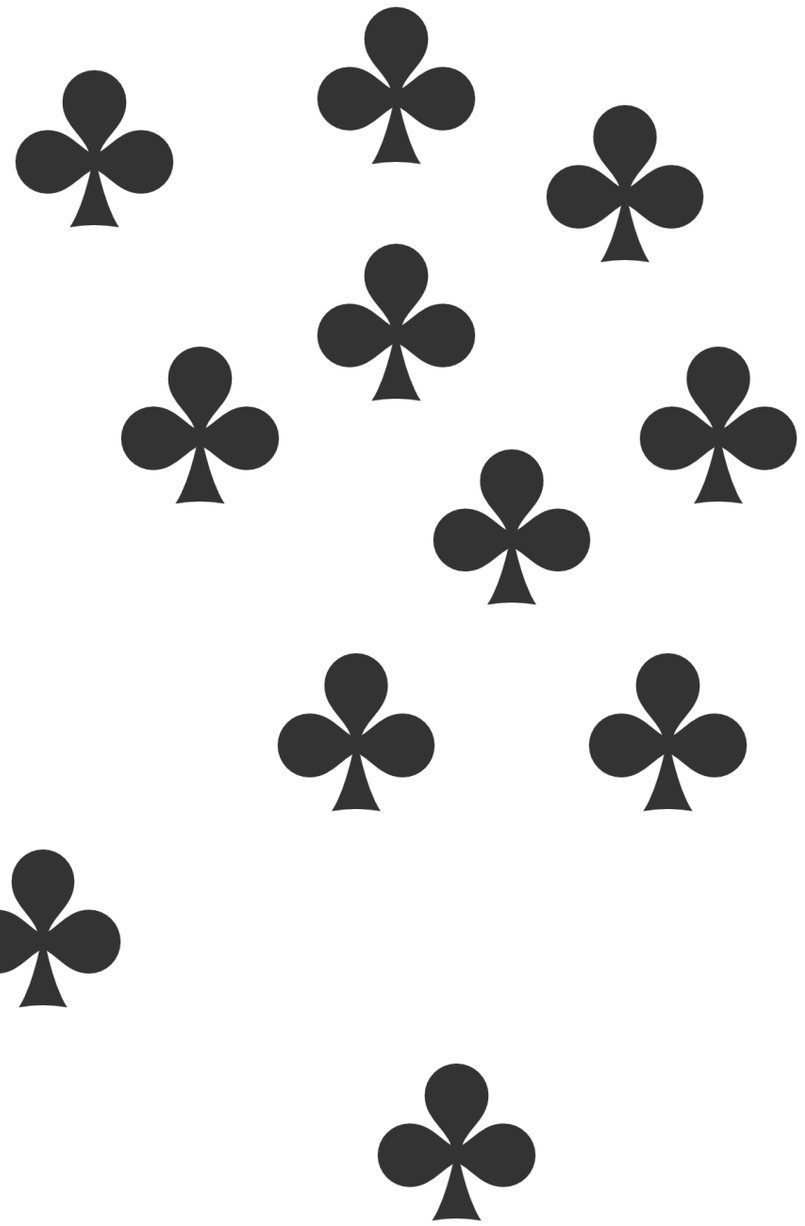
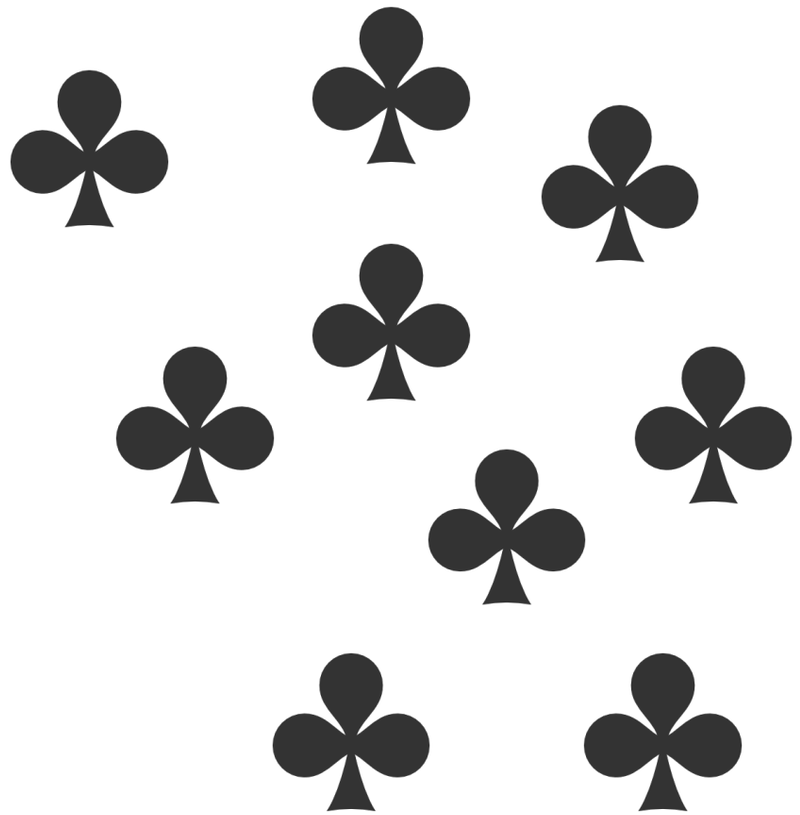
Conceptual content

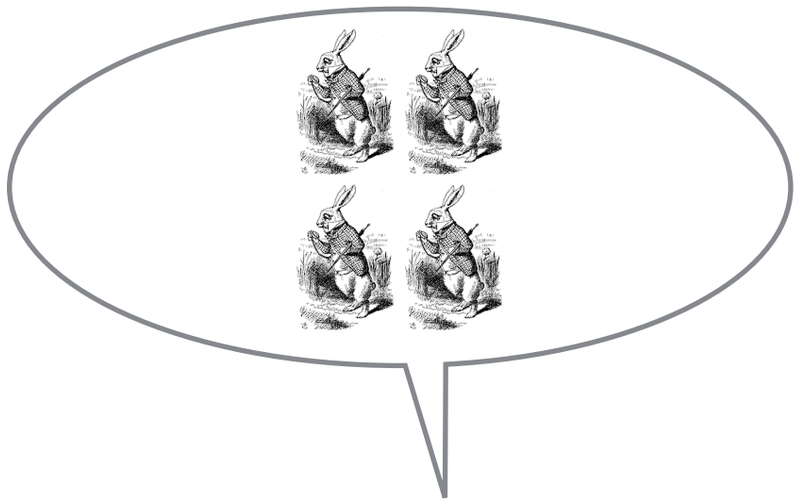
Conceptual content

- core cognition
- cultural input
- individual thinking

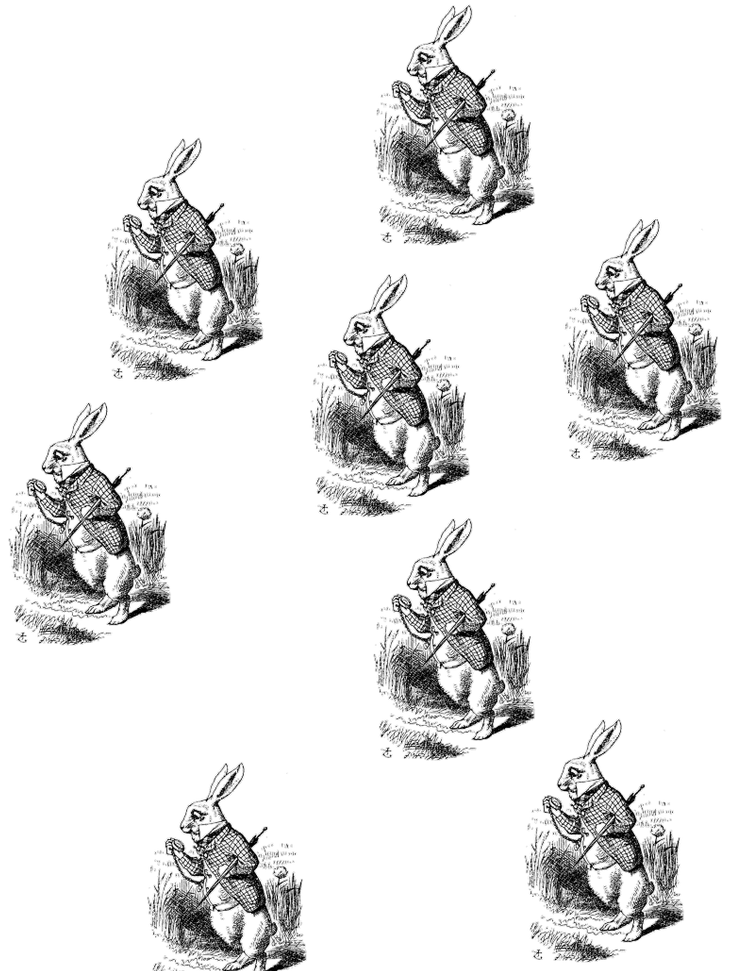


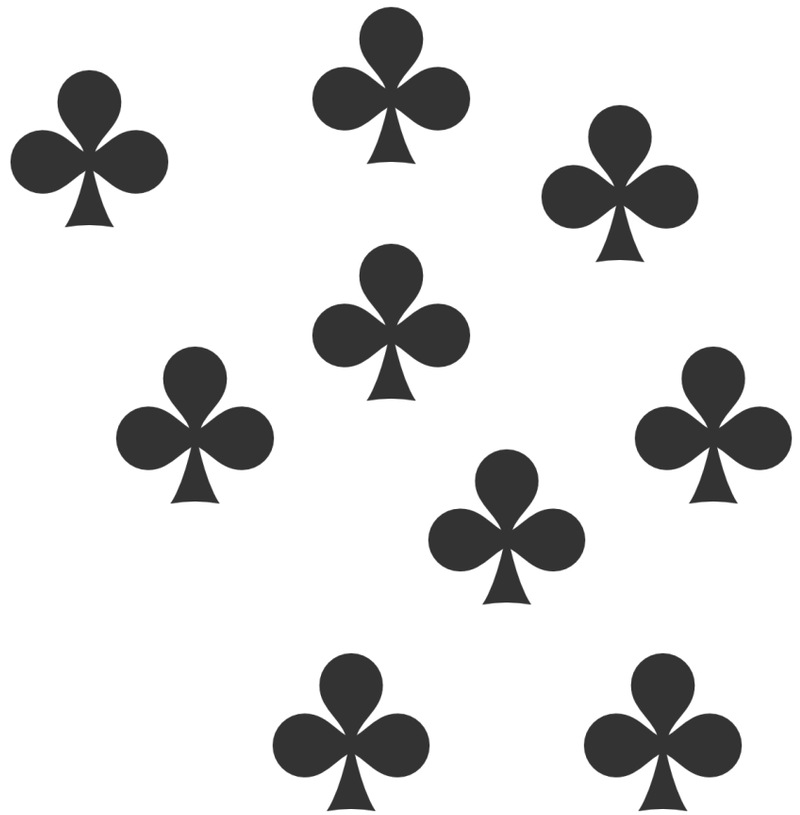


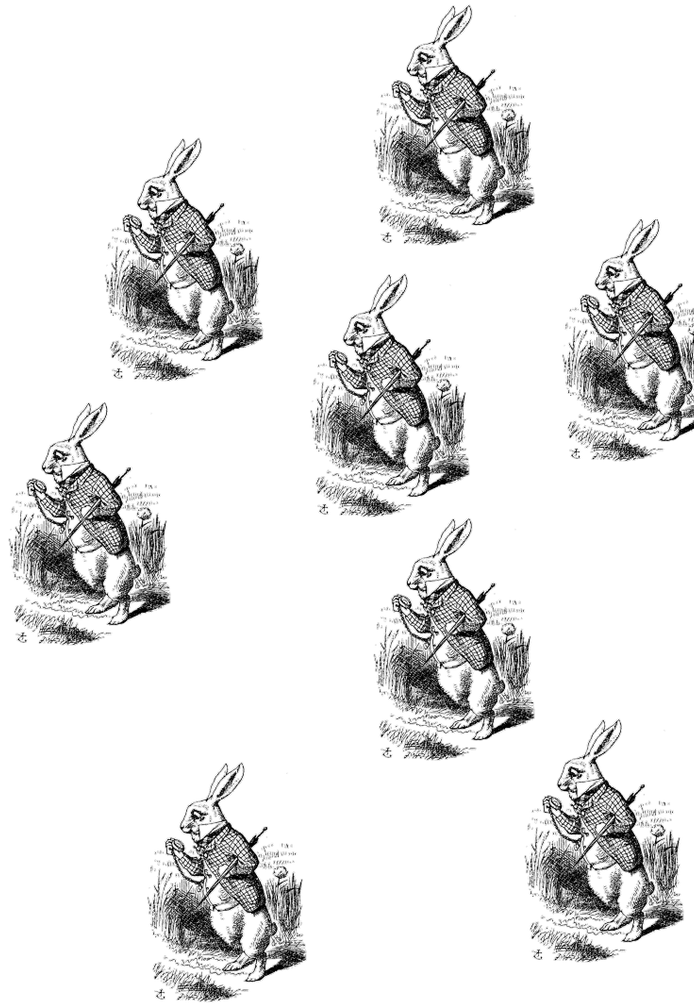


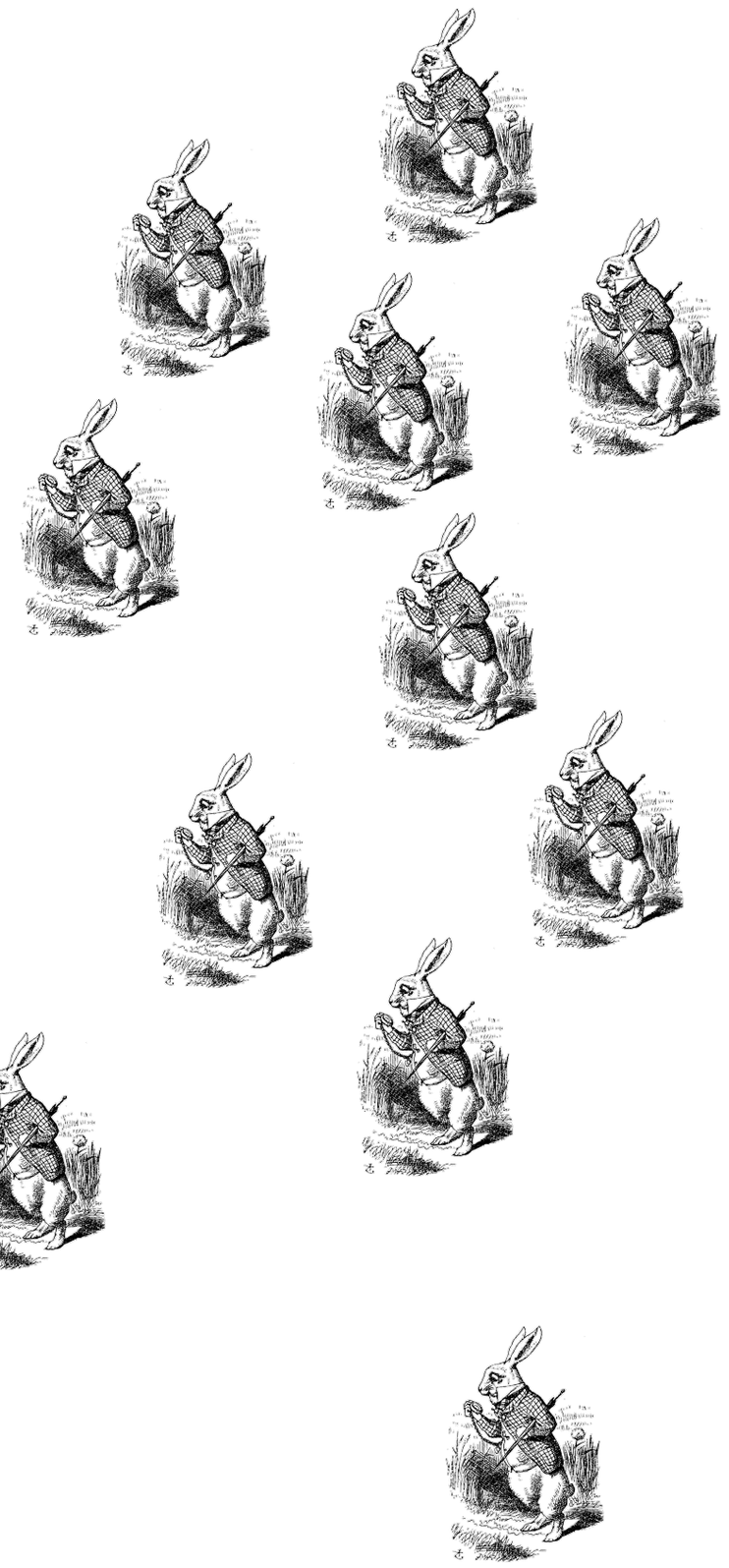


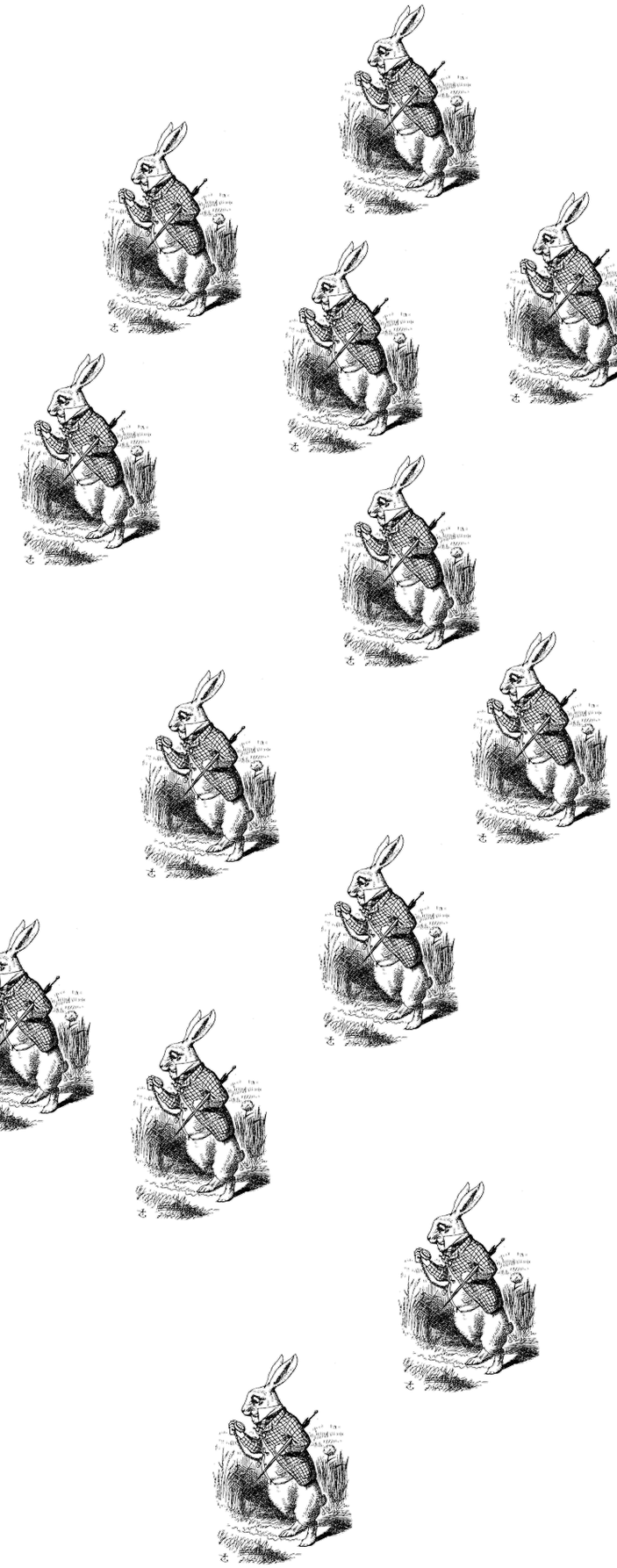
?



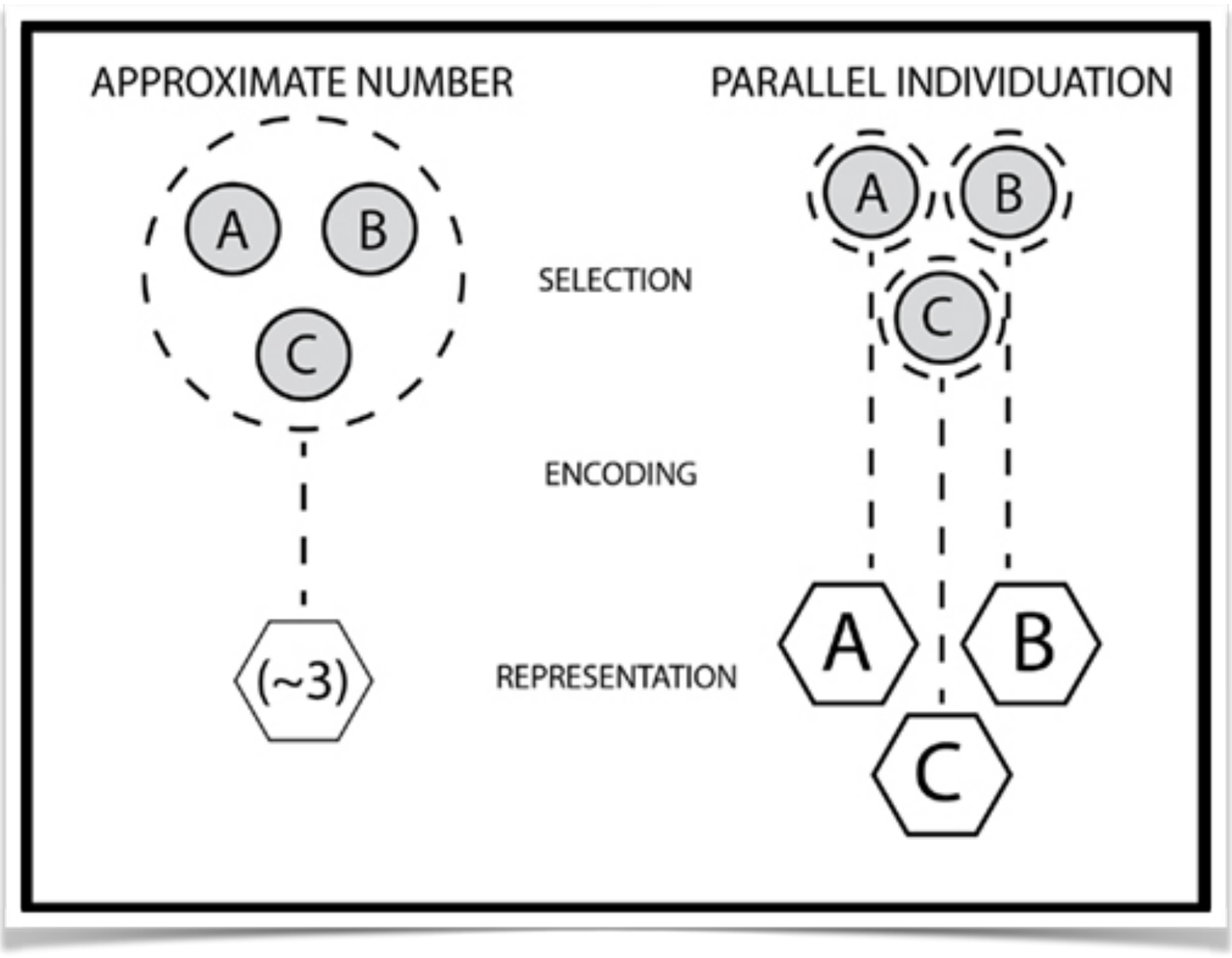






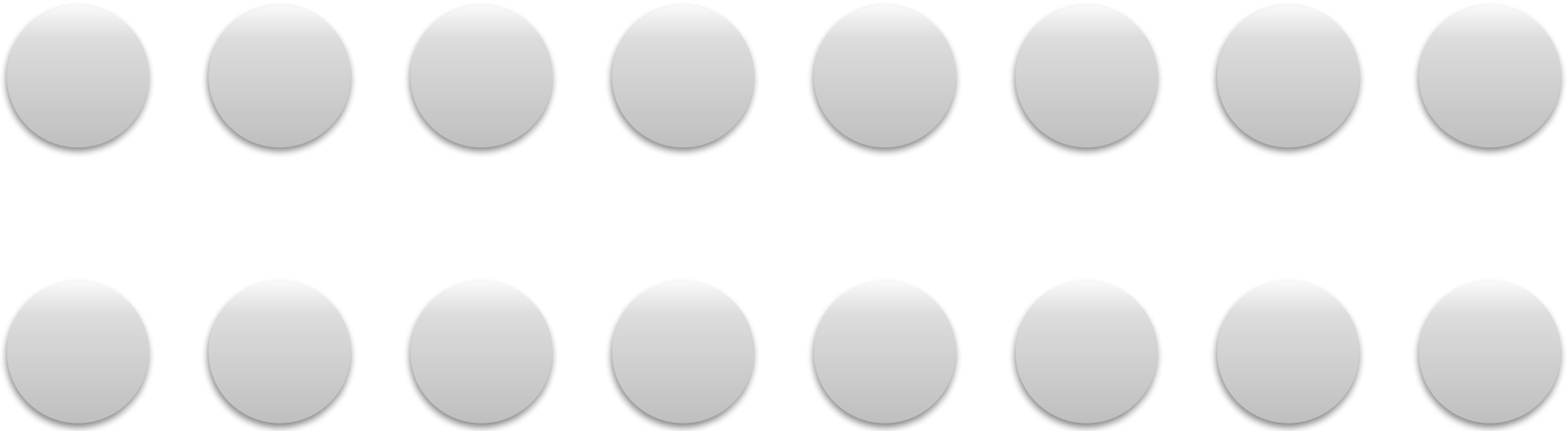


Number-cognition



Hyde 2011

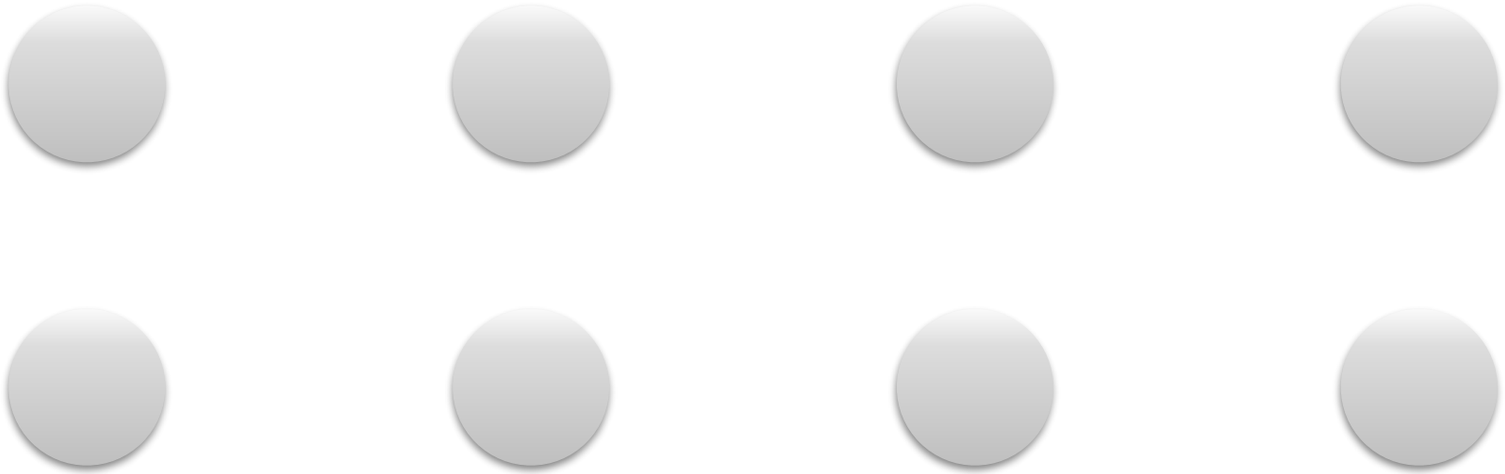
Piaget 1952



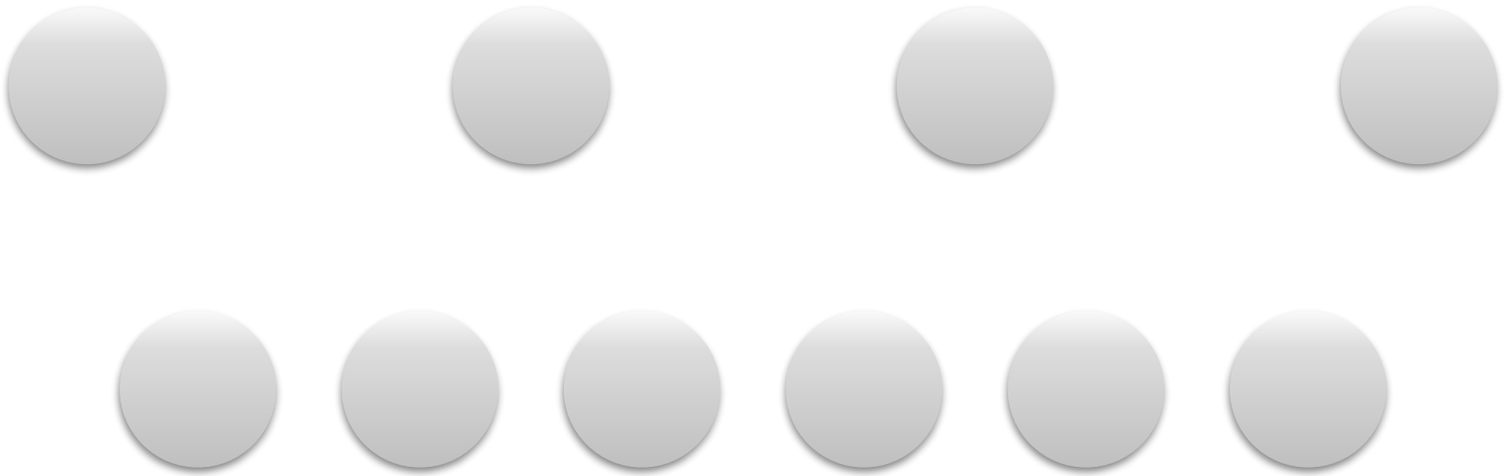
Piaget 1952

Mehler and Bever 1967

Mehler and Bever 1967



Mehler and Bever 1967



Mehler and Bever 1967

First conceptual content

1, 2, 3, 4, 5, 6, 7, 8, ...

arbitrary progression of names

First conceptual content

1, 2, 3, 4, 5, 6, 7, 8,....

subitising

Conceptual enrichment of the first 4 numerals

- **One-knowers:** One-knowers know that ‘one’ means 1, and that all of the other number words mean something different from 1.

- **Two-knowers:** Two-knowers know that ‘one’ means 1 and that ‘two’ means 2, but they make no distinctions among any other numbers.

- **Three-knowers:** Three-knowers understand, in addition, that all numbers apply to discrete entities such as blocks, but not to continuous entities (such as water).

- **Four-knowers:** Four-knowers understand that number words appearing later in the list denote sets with more items.

- **CP-knowers:** understand many things about number that children at earlier levels do not. Only CP- knowers recognize that number words pick out numerosity, as opposed to some other property of sets, such as total summed area or contour length.

Sarnecka’s Lab Irvine

Conceptual content provided by subitizing

Which conceptual content do we get from subitizing?

- Discreteness

First conceptual content

ANS

1, 2, 3, 4, **5, 6, 7, 8,....**

cardinality principle

ANS-acuity

ANS-acuity improves [Halberda & Feigenson 2008]

- 3-years-olds: 2:3; 4-years-olds: 3:4; 5-years-olds: 4:5; 6-years-olds: 6:7; adults: 9:10.

Basing understanding meanings of numeral line (LNS) on ANS-acuity was recently criticised by Negen (2014)

Conceptual content of ANS

Which conceptual content do we get from the Approximate Number Structure?

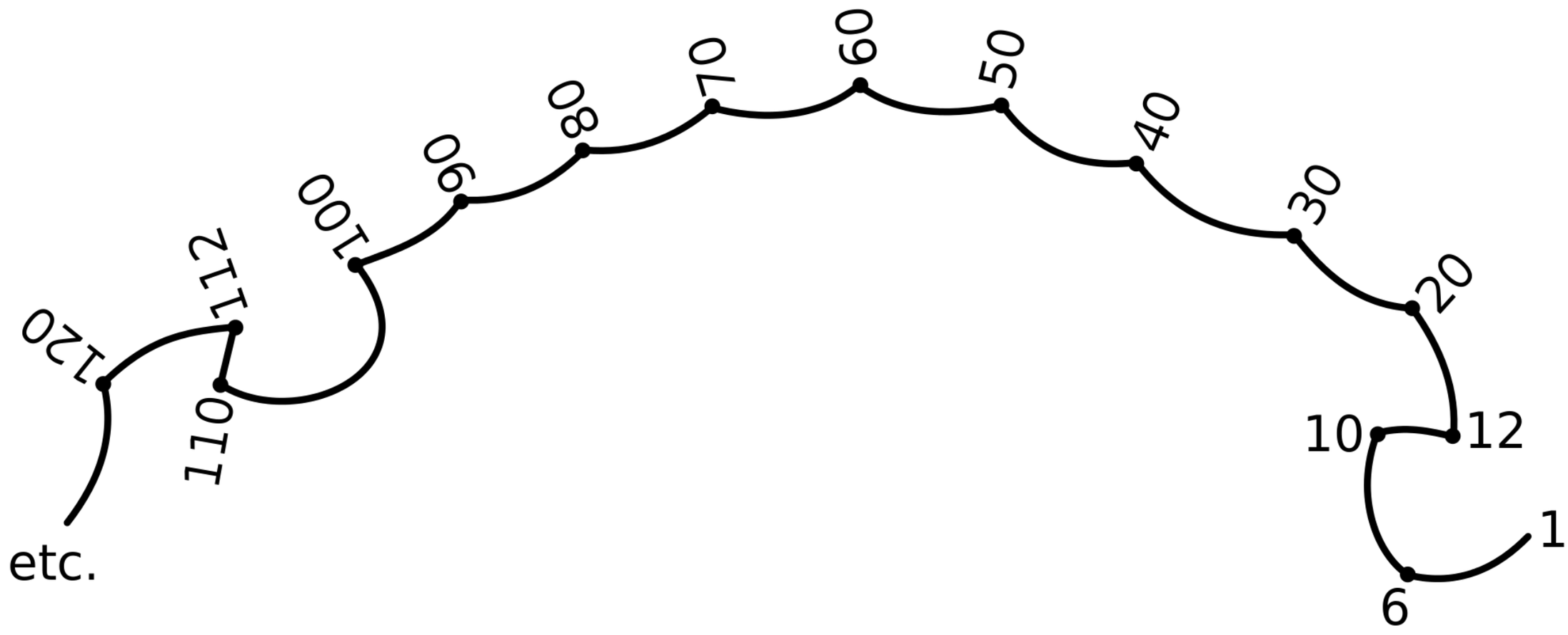
- Reaction to discrete quantities
- Progression of magnitudes

Conceptual content of ANS

Which conceptual content do we get from the Approximate Number Structure?

- Reaction to discrete quantities
- Progression of magnitudes ?

Synesthesia



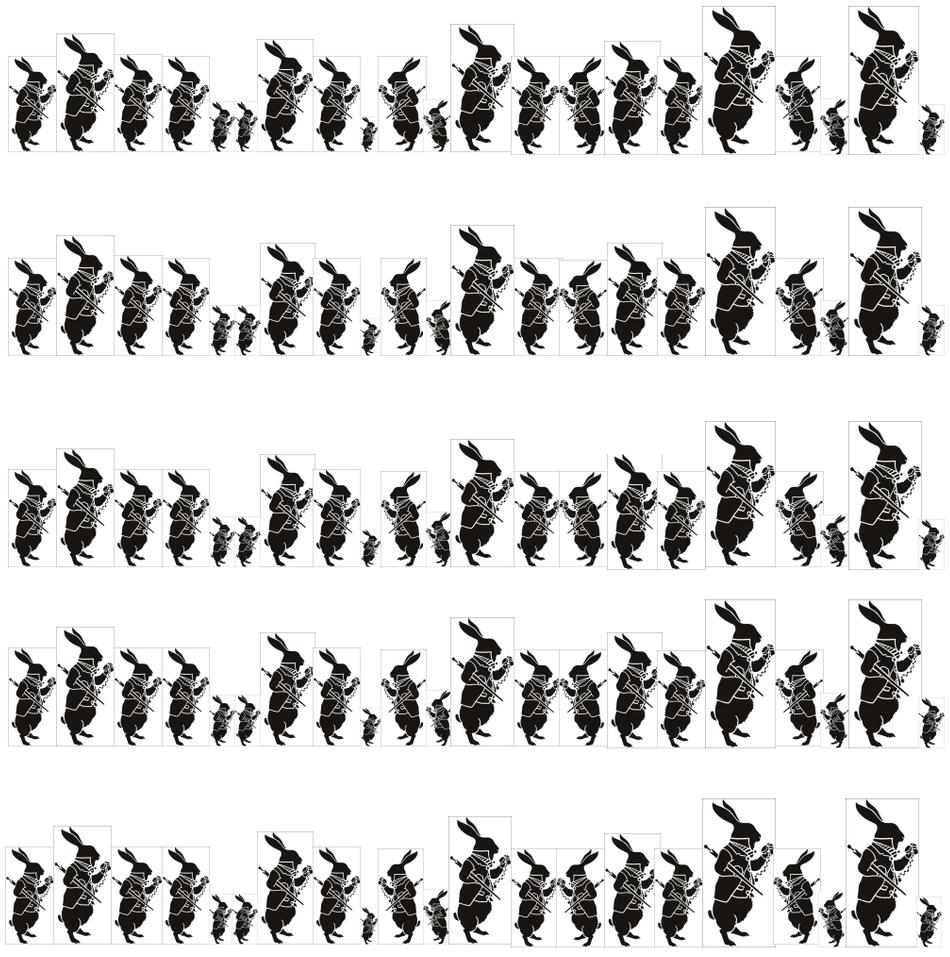
Galton 1881

ANS-calibration

- ANS is also activated by symbolic input

ANS-calibration

- ANS is also activated by symbolic input



ANS-calibration

- ANS is also activated by symbolic input

ANS-calibration

- ANS is also activated by symbolic input

0, 1, 2, 3, 4, …, 51, 52, …, 92, 93, 94, …, 105, …

ANS-calibration

- ANS is also activated by symbolic input

0, 1, 2, 3, 4, …, 51, 52, …, 92, 93, **94**, …, 105, …

ANS-calibration

- ANS is also activated by symbolic input

0, 1, 2, 3, 4, …, 51, 52, …, 92, 93, **94**, …, 105, …

ANS-calibration

- ANS is also activated by symbolic input

0, 1, 2, 3, 4, …, 51, 52, …, 92, 93, **94**, …, 105, …

ANS-calibration

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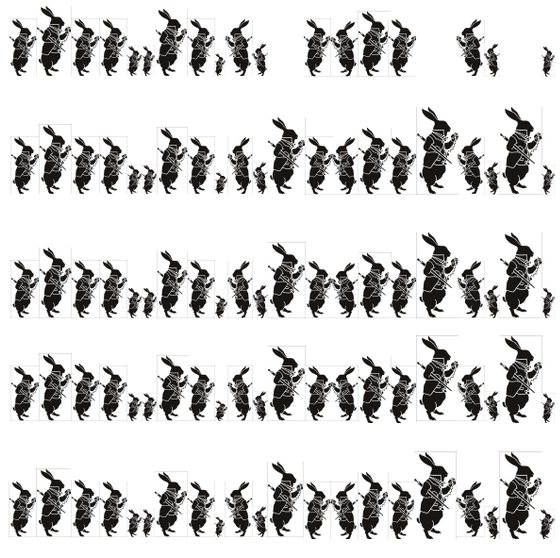


Conceptual content of ANS

Which conceptual content do we get from the Approximate Number Structure?

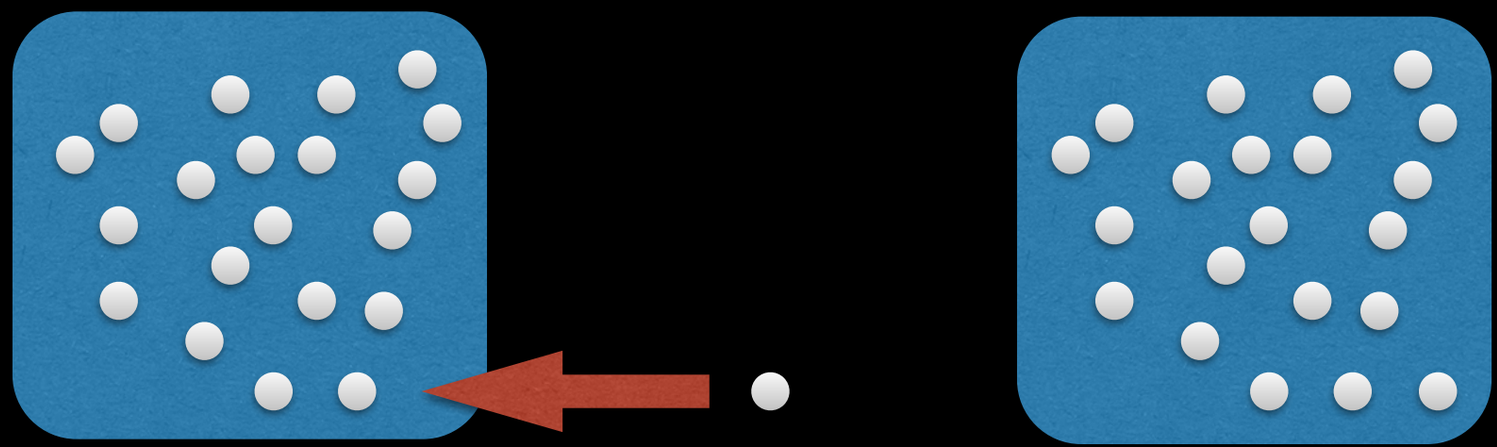
- Reaction to discrete quantities
- Progression of magnitudes

What we do not get is the idea of successor
Not speaking of a regular or a computable successor

Cardinality principle

Understanding that moving forward *one word* in the number-word list means adding *one item* to the named set.

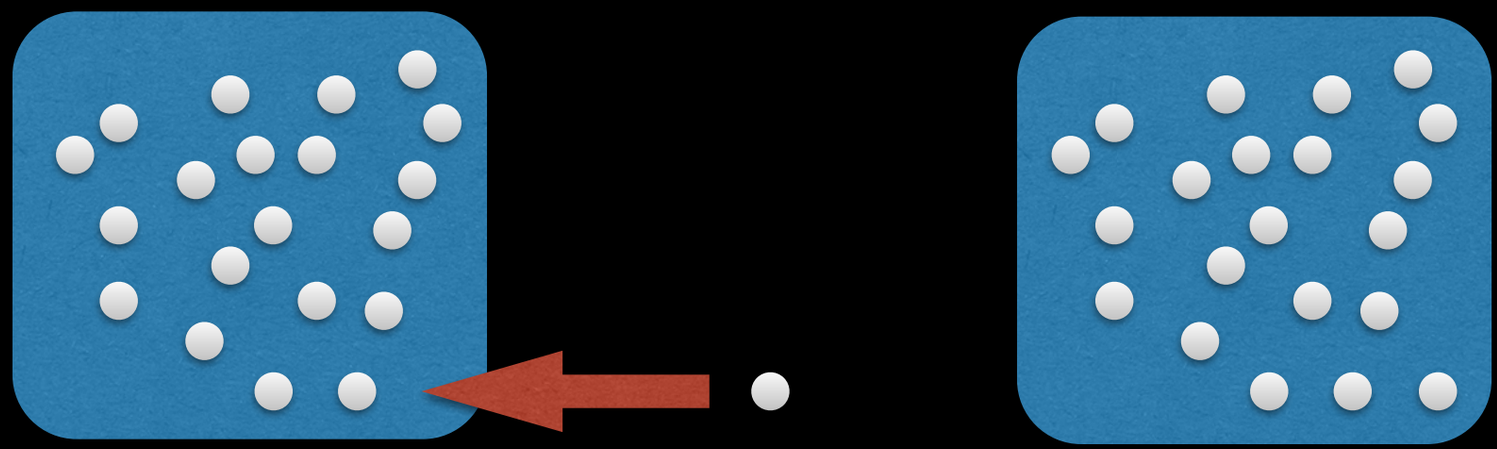
Cardinality principle



Cardinality principle

325 989 871

325 989 872



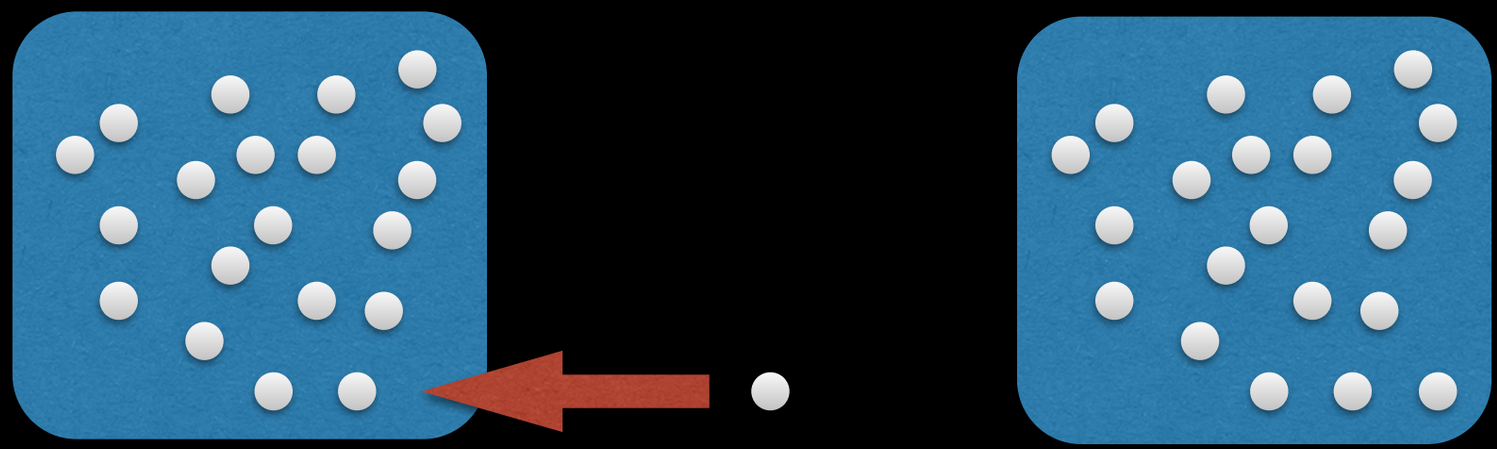
First conceptual content

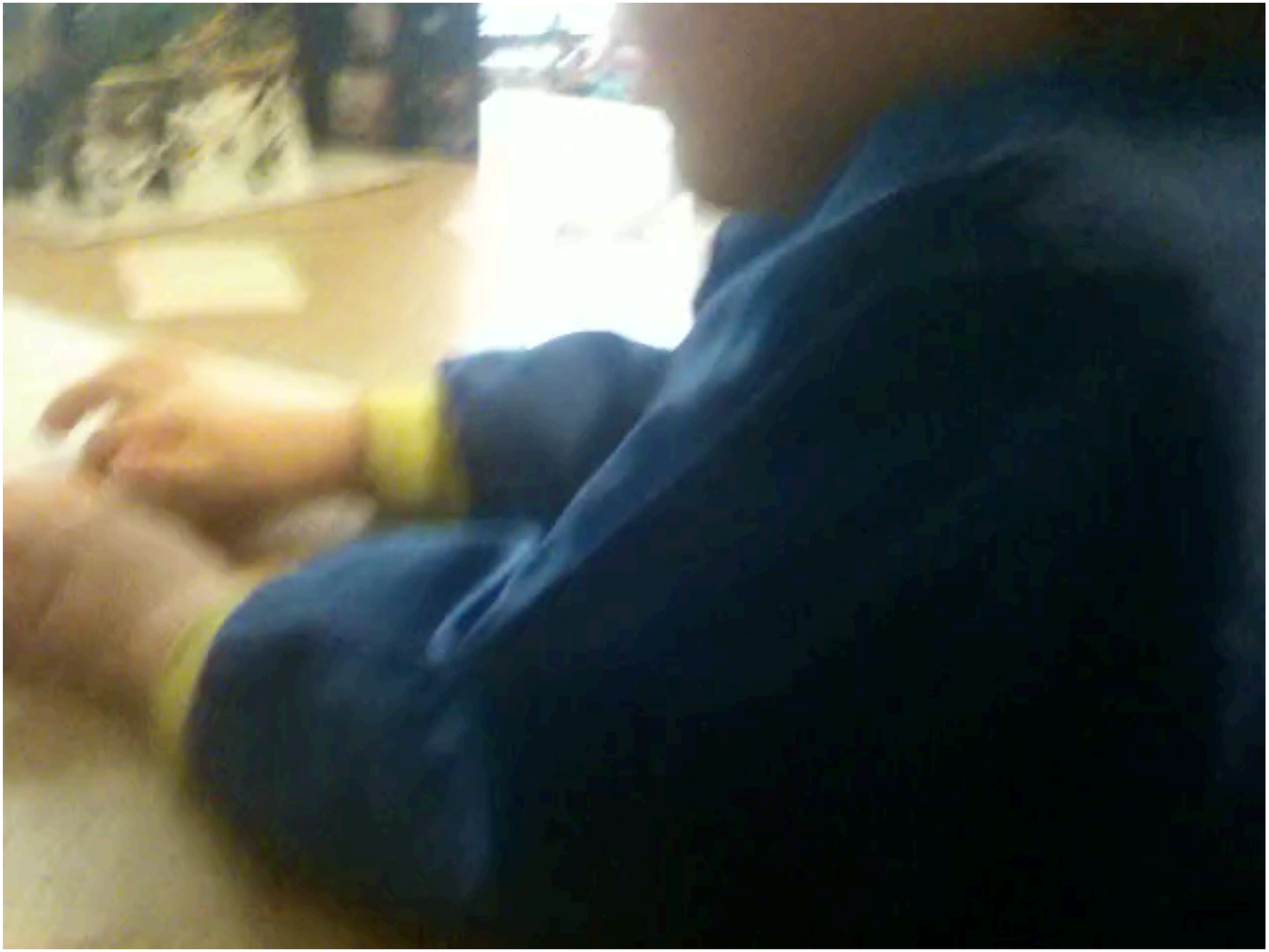
325 989 871

Máximo Pérez

325 989 872

The Railroad





First conceptual content

1, 2, 3, 4, 5, 6, 7, 8,....

subitising

First conceptual content

ANS

1, 2, 3, 4, **5, 6, 7, 8,....**

cardinality principle

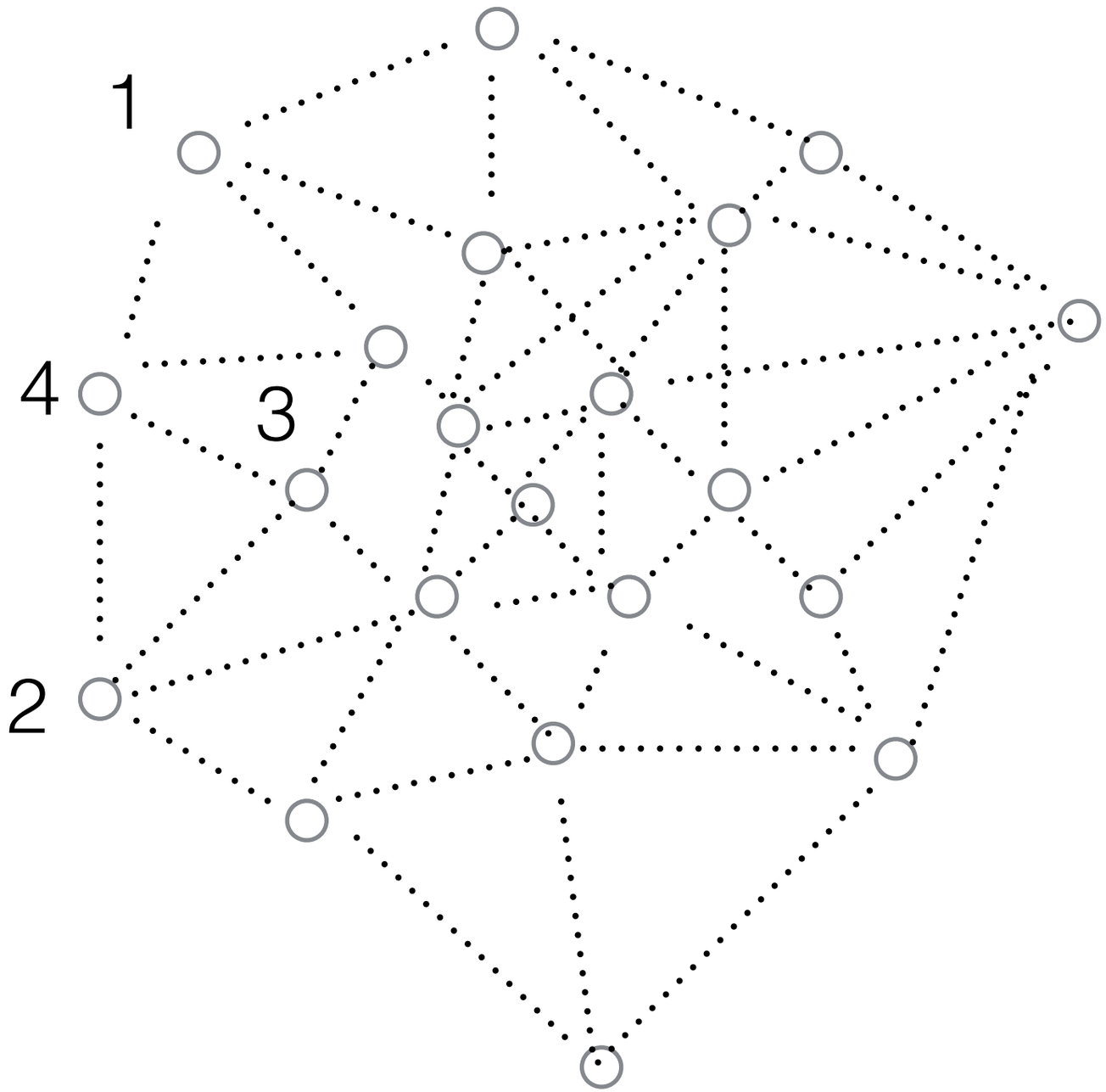
Bootstrapping

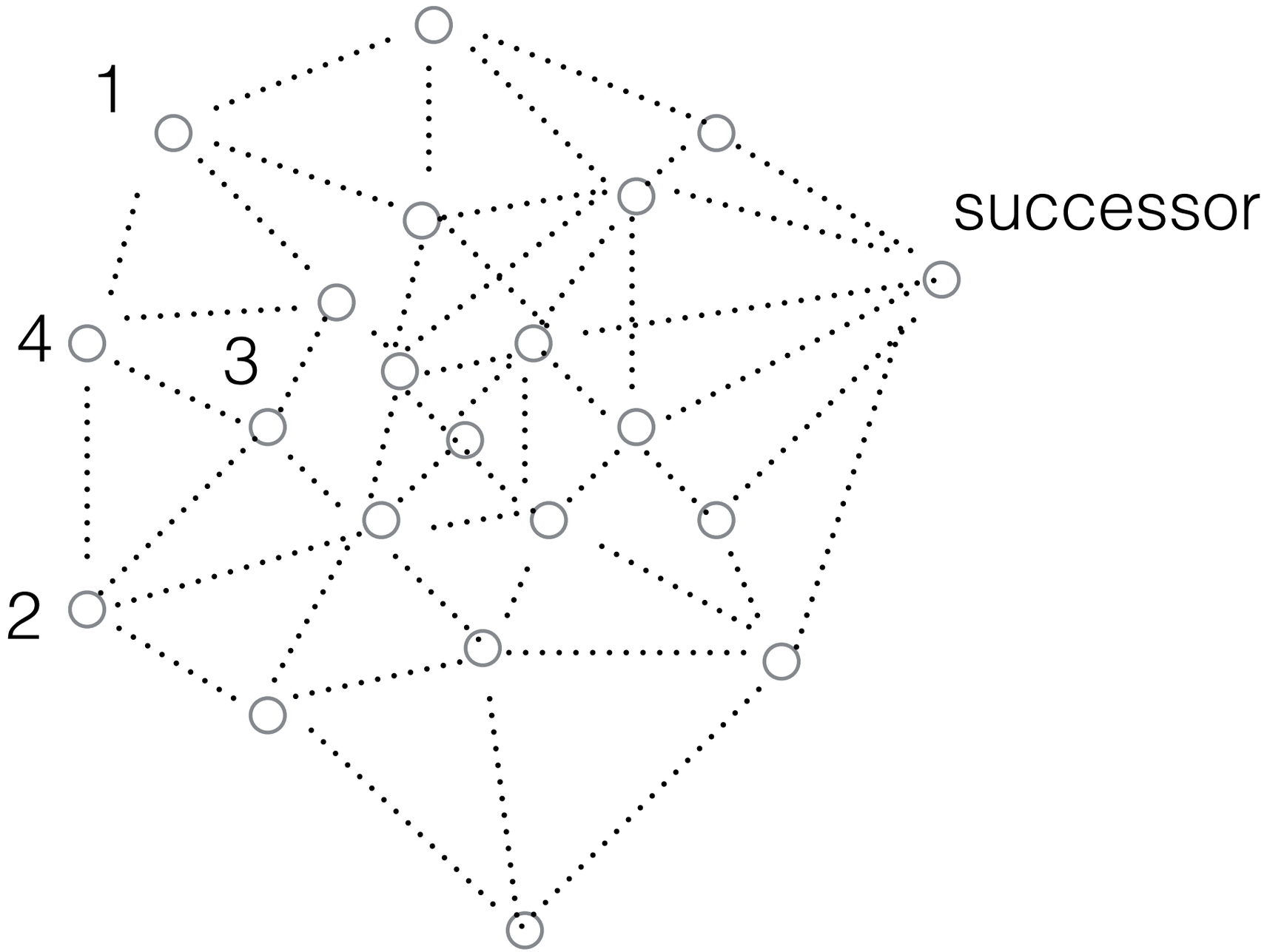
Bootstrapping (Carey 2009)

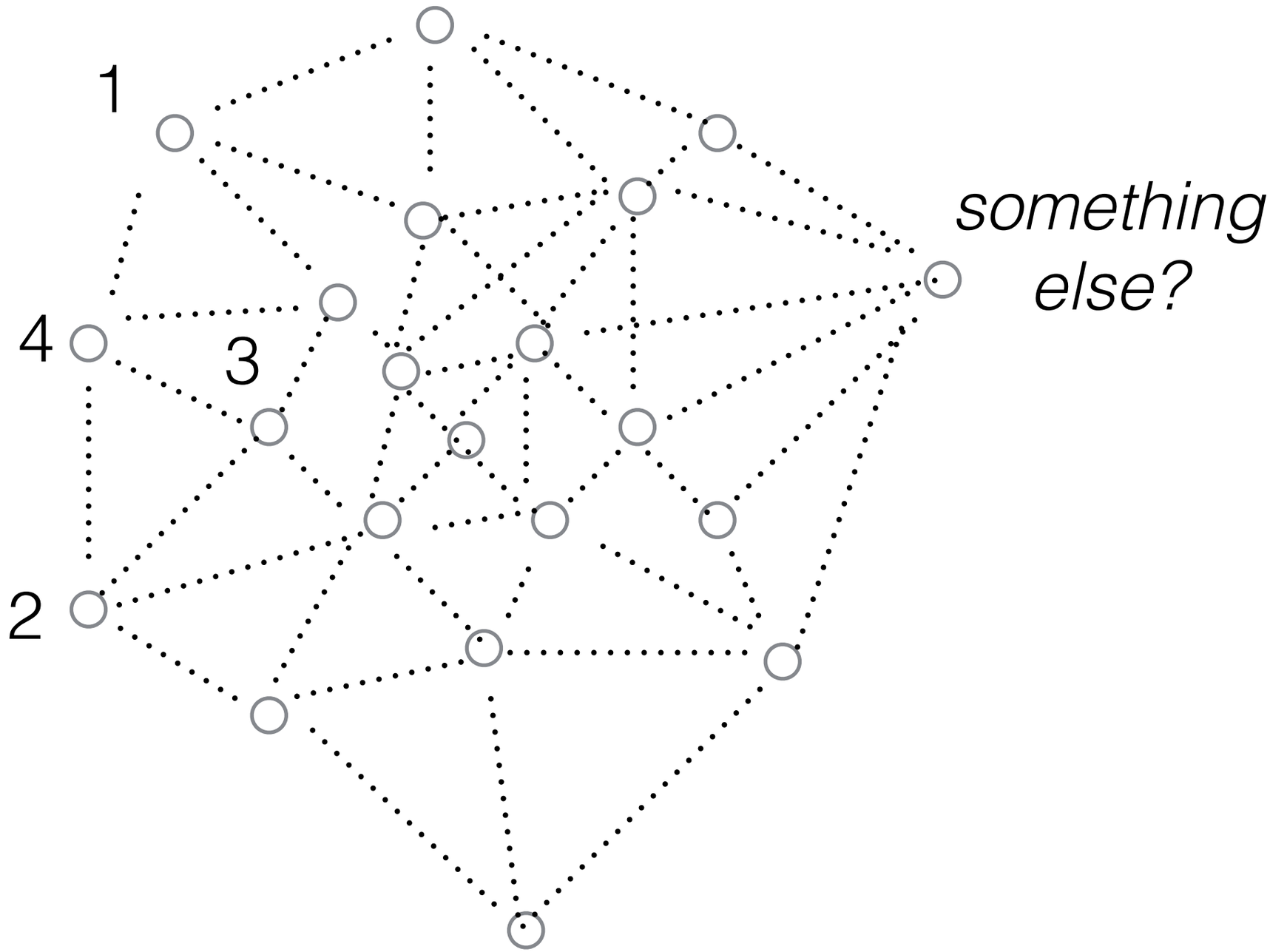
- building a chimney, pressing against the sides to support oneself as one scrambles up it,

- building a ladder, and then kicking the ladder out from under, and

- Neurath's boat, in which one builds a structure to support oneself while already at sea [Quine]


1
4
3
2


1
successor
4
3
2


1
2
3
4
something
else?

Bootstrapping

Two candidates for bootstrapping:

- ANS
- cardinality principle

ANS + CP

- Reaction to discrete quantities
- Progression of magnitudes
- Discrete successor

What we do not get is the idea of regularity

Syntax

Part 2

conceptual content

conceptual content

. . .

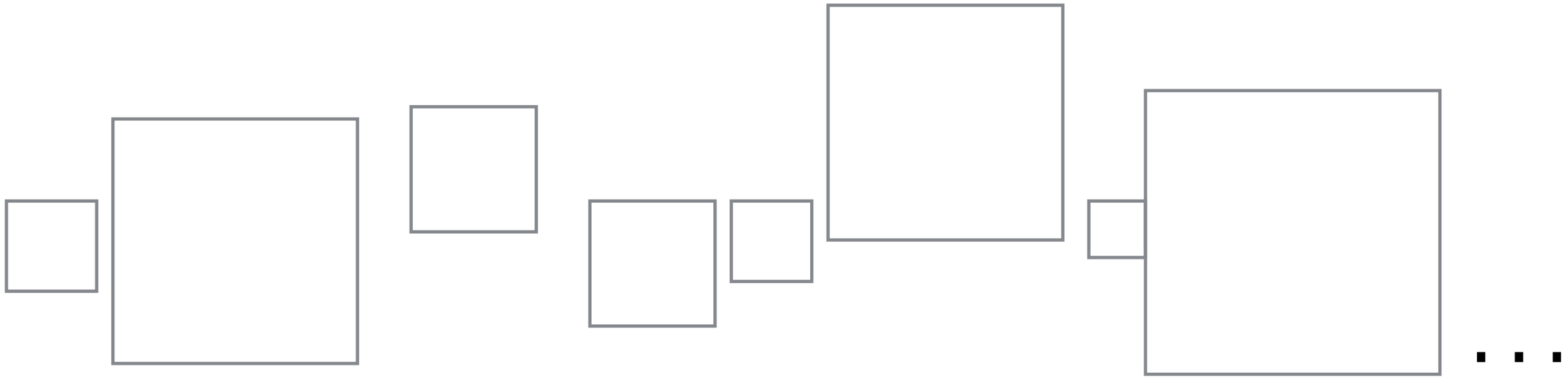
conceptual content

conceptual content

conceptual content

conceptual content

conceptual content










Non-recursive notation

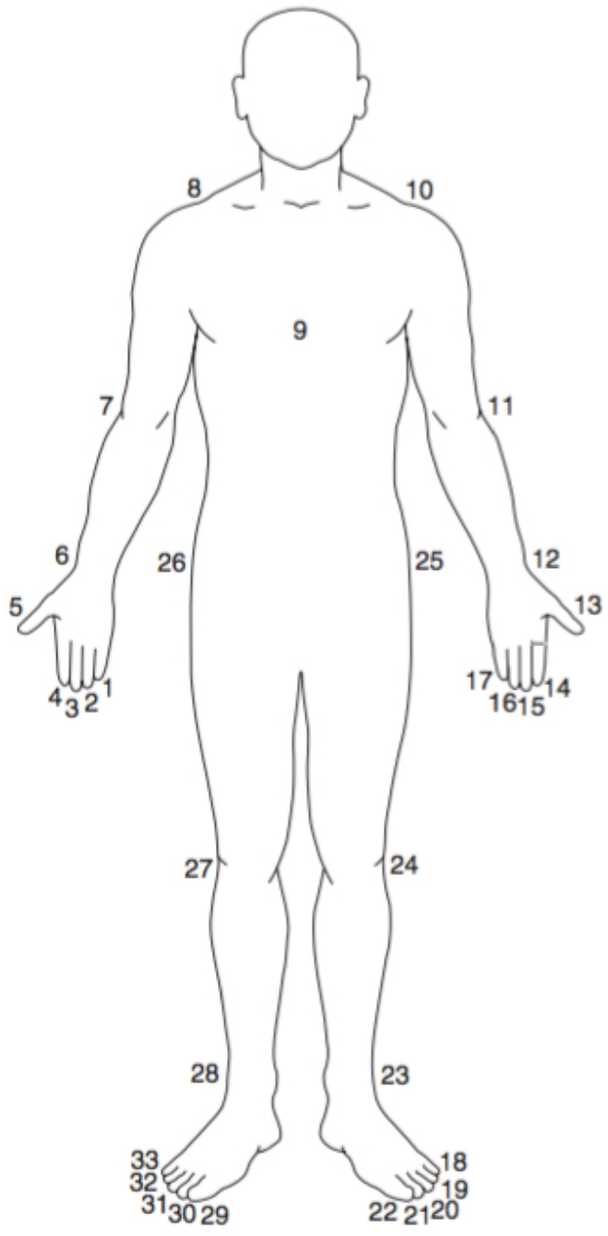
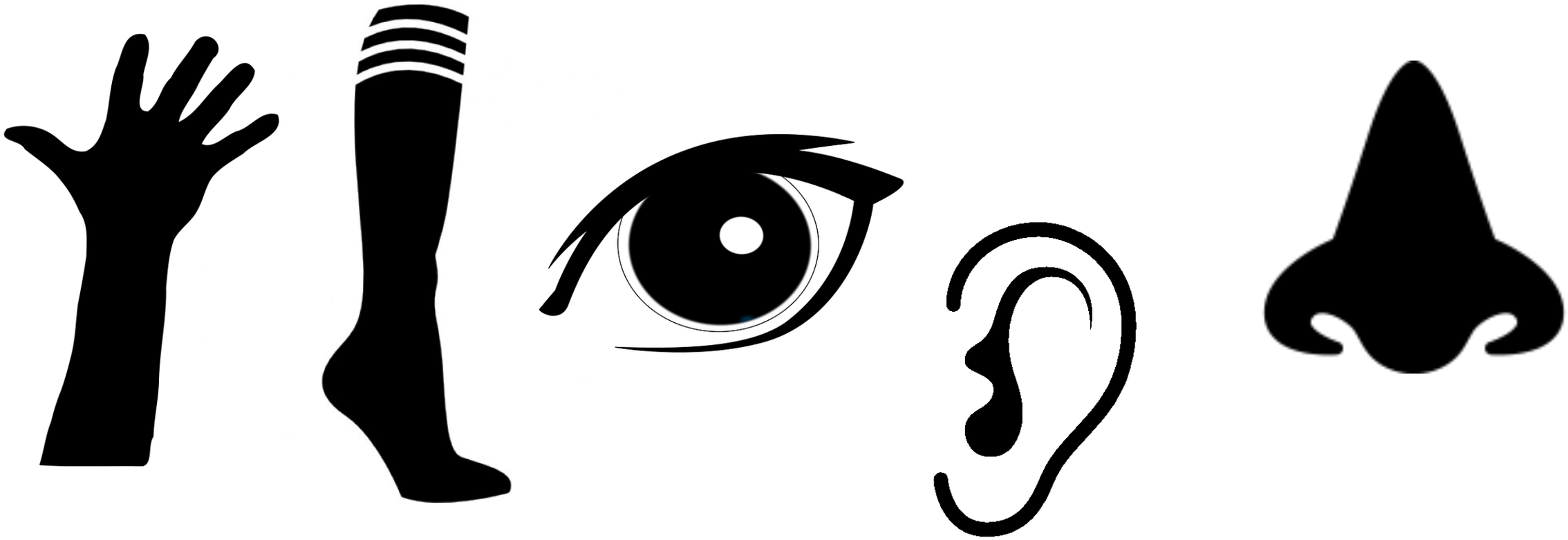


FIGURE 4.1. The natives of the Torres Straight denote numbers by pointing towards a precise part of their body.
(After Ifrah 1998.)

Non-recursive notation



Linguistic Number Line

He told me that in 1886 he had invented a numbering system original with himself, and that within a very few days he had passed the twenty-four thousand mark. (…) Instead of seven thousand thirteen (7013), he would say, for instance, “Máximo Pérez”; instead of seven thousand fourteen (7014), “the railroad”; other numbers were “Luis Melián Lafinur,” “Olimar,” “sulfur,” “clubs,” “the whale,” “gas,” “a stewpot,” “Napoleon,””Agustín de Vedia.” Instead of five hundred (500), he said “nine.” Every word had a particular figure attached to it, a sort of marker; the later ones were extremely complicated…. I tried to explain to Funes that his rhapsody of unconnected words was exactly the opposite of a number system. I told him that when one said “365” one said “three hundreds, six tens, and five ones,” a breakdown impossible with the "numbers" Nigger Timoteo or a ponchoful of meat. Funes either could not or would not understand me. (Borges, Funes the Memorious).

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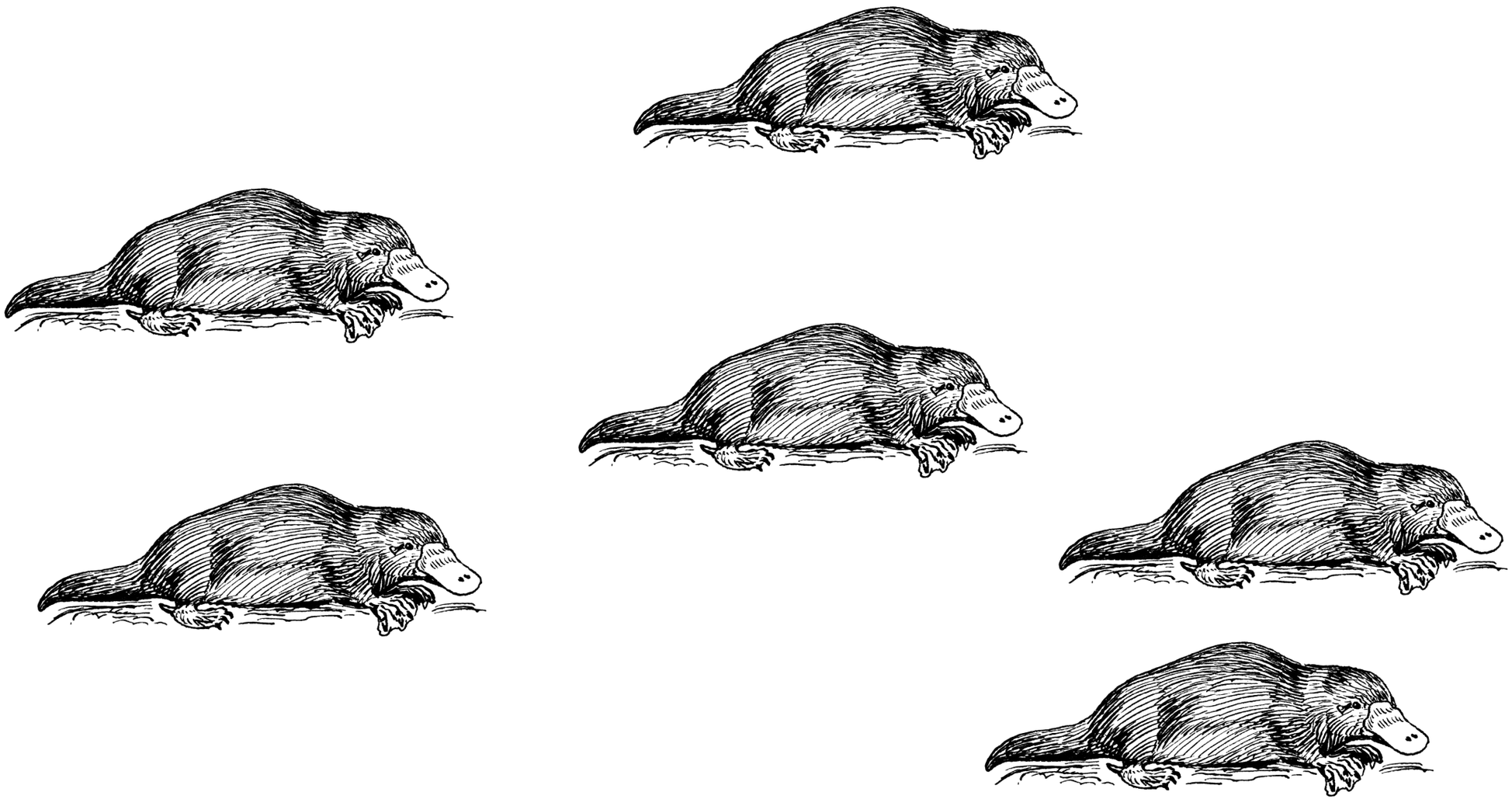
Benacerraf 1996



Which conceptual knowledge needs a Funes-child?

ANS + CP

- Reaction to discrete quantities
- Progression of magnitudes
- Discrete successor

What is missing?

ANS + CP

- Reaction to discrete quantities
- Progression of magnitudes
- Discrete successor

What we do not get is the idea of regularity

Regularity

- Hypothesis: beat-induction

Conclusions

Conceptual content from specific conceptual input (cognitive or not)

1. subitizing gives us the concept of discrete quantity;
2. the ANS gives the idea that numerals form a progression;
3. finally, the sensitivity to rhythm ensures that the progression is regular.

Enculturation 2.0

- Carnap's Explications

- Language develops on the *semantical* level (meanings changes, new concepts, new uses of old concepts), but also on *syntactic* level.

- Hauser, Chomsky et al. 2002 support the idea that there is a specific part of the human language (a so called deep structure): *recursivity.* In particular, in the case of natural numbers it is particularly stacking.

- Digitalisation

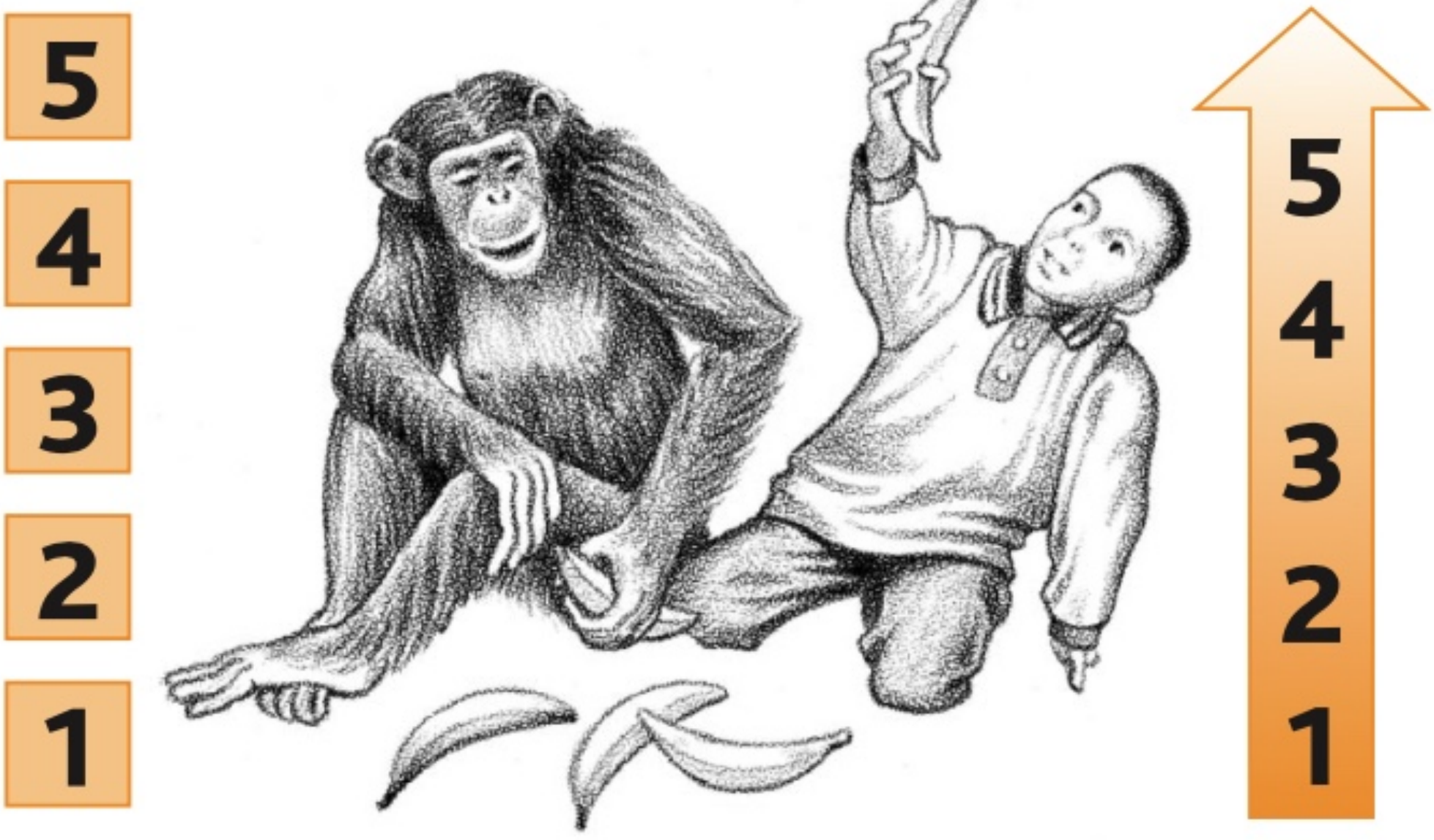


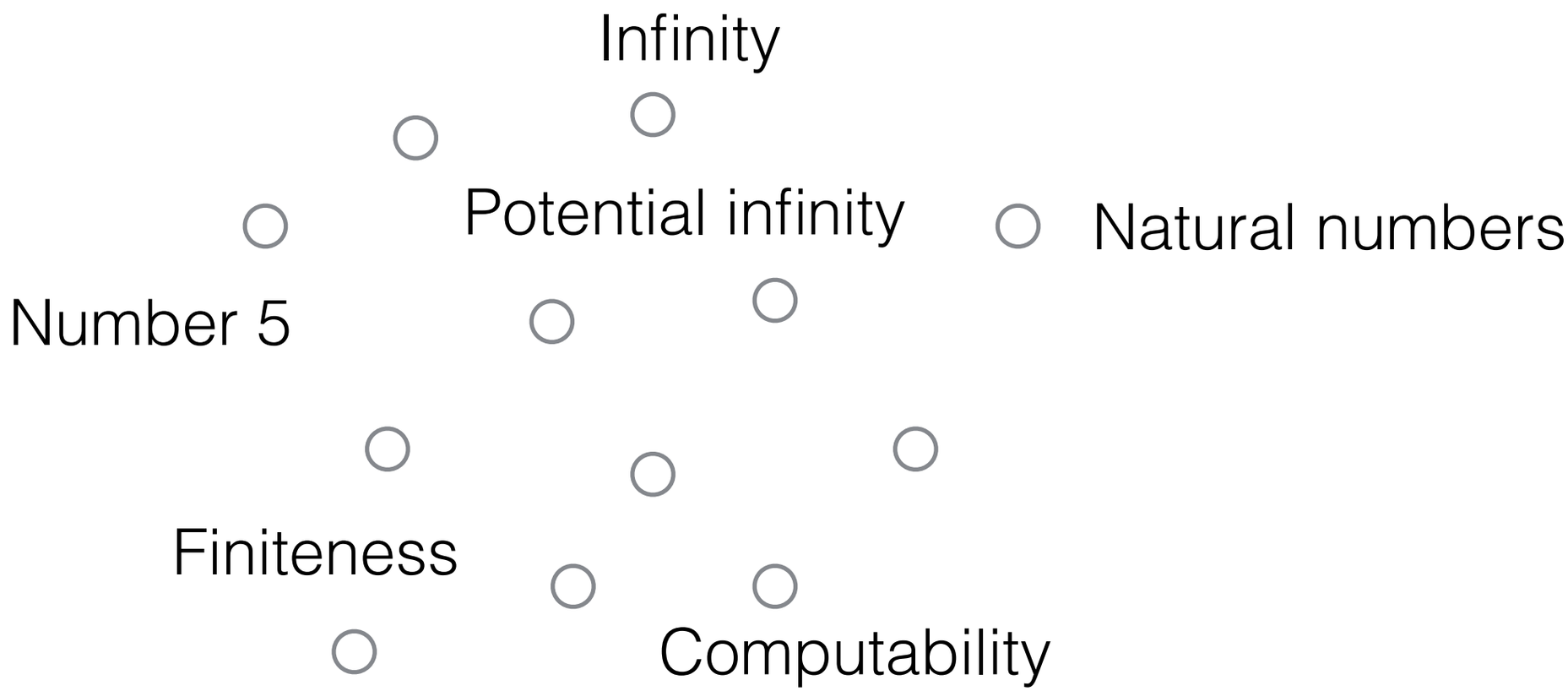
Fig. 5. Human and nonhuman animals exhibit the capacity to compute numerosities, including small precise number quantification and large approximate number estimation. Humans may be unique, however, in the ability to show open-ended, precise quantificational skills with large numbers, including the integer count list. In parallel with the faculty of language, our capacity for number relies on a recursive computation. [Illustration: John Yanson]

Frege's Constraint

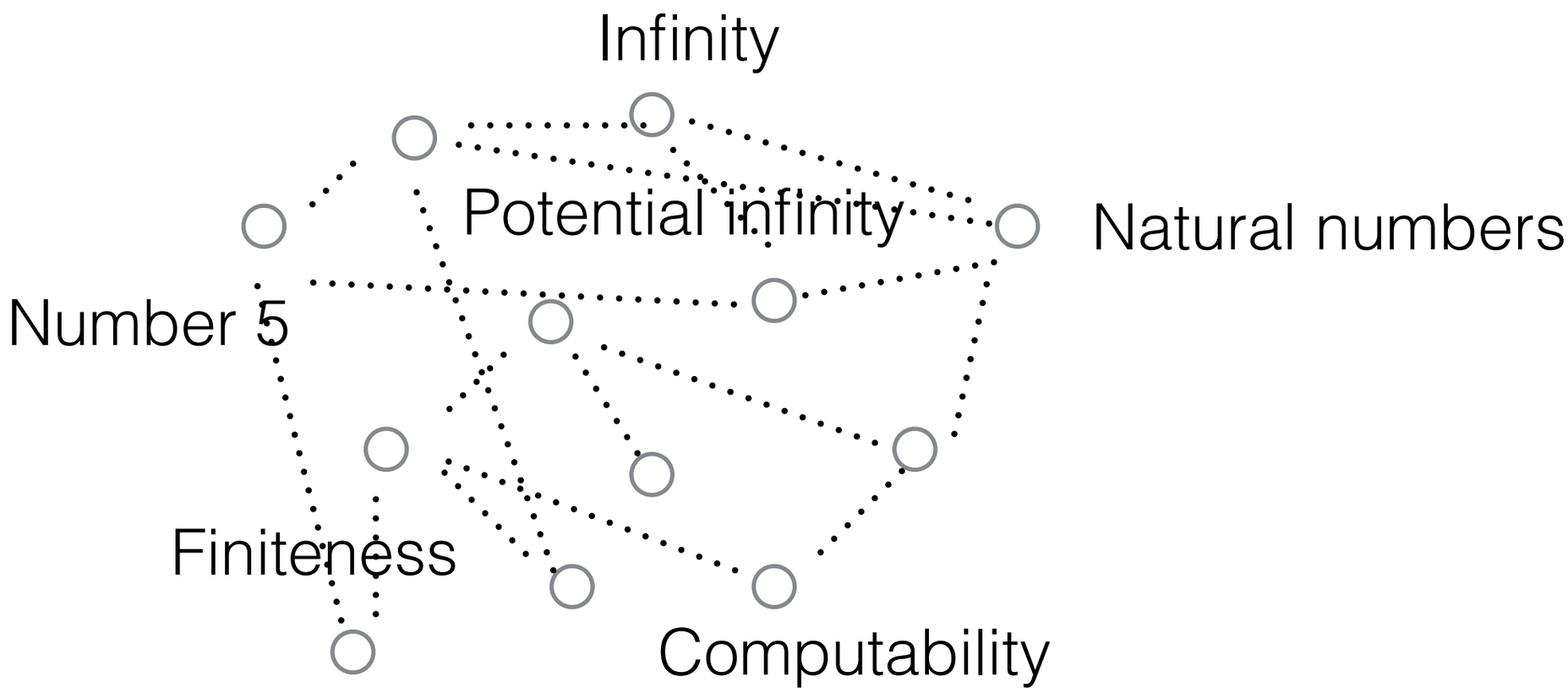
Foundations (axiomatic system or set of first principles) of a mathematical theory are satisfactory when they account for applications of the entities forming the intended model of this theory

Candidate constraints:
- cardinality
- computability
- omega-structure

Philosophy of arithmetic



Philosophy of arithmetic



Benacerraf 1965
Horsten 2012

There are multiple ways in which we can construct the concept of natural number

Extended Frege's Constraint

in addition to the fundamental application encoded at the foundational level, all the other natural properties or applications of the entities forming the intended interpretation should be as well interpreted in the intended manner

Duncan et al. 2007

Early mathematical knowledge is strongly correlated with the later mathematical achievements

Optimality

Which concepts shall we learn first, in order to get the number line mature?

“Can you of Addition?” the White Queen asked. “What’s one and one and one and one and one and one and one and one and one and one?”

“I don’t know,” said Alice. “I lost count.”

“She can’t do Addition,” the Red Queen interrupted.

Lewis Carroll, Through the Looking Glass