The Intended Model of Arithmetic.
An Argument from Tennenbaum's Theorem

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Abstract. We present an argument that allows to determine the intended model of arithmetic using some cognitive assumptions and the assumptions on the structure of natural numbers. Those assumptions are as follows: the psychological version of the Church thesis, computability of addition and multiplication and first order induction. We justify the thesis that the notion of natural number is determined by any recursive ω-model of PA up to recursive isomorphism.

1 Introduction

In this paper we justify the following thesis: our notion of natural number is determined by any recursive ω-model of PA up to recursive isomorphism.

It is a well-known fact that first order Peano arithmetic has infinitely many different models. Most of them are called *non-standard* and only one class of isomorphic models is considered as *standard*.[4] We call a model of arithmetic standard if its ordering is of the type ω. We used to consider that standard model of arithmetic as the one that reflects our intuitions about natural numbers adequately. A model that reflects our intuitions adequately is called *intended*.

In this paper we want to answer the following questions: why is one of the interpretations of the Peano axioms distinguished among so many others? Are standard models really intended models?

It is important to notice, that we make a distinction between an *intended* model and a *standard* model of arithmetic. The second notion is well known in metamathematics of arithmetic. The intended model is a model that satisfies intuitions concerning natural numbers. These two concepts were often identified even though the the standard model of PA is a well defined metamathematical notion while the intended model refers rather to our intuitions. In what follows, we postulate a restriction of the class of intended models to a proper subclass of standard models.

Using cognitive assumptions we argue that the intended model should be recursive (computable). This fact, together with the first order induction principle, determines a subclass of standard models. Similar arguments were developed in [HH05] in the context of Benaceraff analysis of the standard model. We differ from them in putting the main stress on the cognitive nature of our assumptions.

2 A Structuralist Case. What Is True about Natural Numbers?

Benacerraf in [Ben65] presented an analysis of the notion of natural numbers. He argued that any set of objects with the ω–type ordering can be a model for arithmetic. He claimed

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[4] Of course, we count here models of PA up to isomorphism.

that it is not important which objects play the role of natural numbers. The only things which are important are relations between these objects. This standpoint was further-taken by structuralism. Indeed, as long as we are interested in the question: what is true about natural numbers? it is sufficient to consider any model from the class of isomorphic models since isomorphic models have the same theory. And indeed, all models for arithmetic with the ω–type ordering are isomorphic.

Thus, structuralism gives a good answer to the question: what is true about natural numbers? Additionally, an advantage of this approach is that it avoids the ontological problems like: "what is 1?"

The well known critics of structuralism are:

1. A characterization of the ω-sequence needs logic stronger than first order. This problem involves discussion about the first order thesis.[5]
2. Standard models can differ in a nontrivial way, for example not all of them are recursive. We consider this problem later in our paper.
3. The claim that natural numbers are objects of the ω-sequence is not justified. We explain *why* it has to be the ω-sequence.

Our approach continues the lines started in the paper by Halbach and Horsten, [HH05]. They observed there that if we want to point out a subclass of models of Peano arithmetic with ω–type ordering it suffices to require that the operations of addition and multiplication in these models are recursive. So, the notion of the standard model with the ω–type ordering can be explained by requiring that the model should be recursive. We differ from Halbach and Horsten in putting the main stress on this assumption about recursiveness and carefully analyzing it. What we say is that it is exactly this assumption that should be the fundamental feature of the intended model of natural numbers.

3   What Natural Numbers May Be Used for?

We differ from structuralists in the fact that we do not take the property of ω–type ordering as constituting natural numbers. Instead of searching for structural properties of natural numbers we begin with the questions how we learn and how we use natural numbers. In consequence, we consider that the basic feature of natural numbers is that we can count using them. Thus, similarly to structuralists we are not interested in the questions like: "what is 1?" It is not important what is 1 but rather whether we can count using a given representation of 1. Contrary to structuralists, we consider that the basic property of natural numbers is the possibility to use them to count. This property should play the decisive role in deciding whether a given model for arithmetic may be considered as being intended or not.

We learn what are natural numbers while we learn to count. Consequently, if a given model for arithmetic is intended, that is reflecting our intuitions about natural numbers, it should be such that one can perform basic arithmetical operations (addition and multiplication) on elements of this models (numbers from this model).

4   Church's Thesis

In order to give the mathematical meaning to the last sentence of the previous section we use the psychological version of Church's thesis.

Let us introduce two notions: the notion of computability and the notion of recursivity. The notion of computability is an informal notion that refers to some class of functions so-called "effectively calculable" functions. In other words, it refers to a class of functions defined on

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[5] See for example: [Sha91].

some fixed space (e.g. the space of finite words of a given finite alphabet) for which there exists a method, determined by unambiguous instructions, of associating to each set of arguments $x_1, x_2, \ldots, x_n$ the value $f(x_1, x_2, \ldots, x_n)$ in a finite number of steps. The second notion: recursivity, refers to the class of function mathematically defined. For the present purposes we may use e.g. the notions of Turing machine as a computing device and we may define the class of recursive functions as those which are computed by some Turing machine.

We want to stress here that we treat computability as a basic notion. Moreover, we do not restrict computability to a one fixed domain of objects. We think rather of computability as an activity which may be related to any set of objects satisfying some basic assumptions.[6]

In the literature we find two versions of Church's thesis. The first one, *ontological*, says what we can be computed in the real world: all computable functions are recursive functions. The second version, *psychological*, says that cognitive capacities of humans contain at most recursive functions. Since in this paper we are mainly interested in the psychological version of Church's thesis, let us formulate it properly (for a detailed analysis see [MW04]

Thesis 1 (The psychological version of Church's thesis) *Any property that human can compute can be also computed by Turing machines.*

This thesis gives an upper bound on what we may compute. Notice also that from this assumption it follows that in order to distinguish a class of intended models we cannot identify any two isomorphic models, but at most recursively isomorphic models.[7] Indeed, if two models are isomorphic but not recursively isomorphic then their equivalence can be regard as beyond our cognitive capacities.

The falsehood of the psychological thesis has as a consequence the falsehood of the ontological thesis but the inverse may not hold. This is so because of a situation, a priori possible, when there are some nonrecursive functions which potentially (without any time nor space limitation) could be computed by a physical device but not by a human mind. Since we analyze here the class of intended models we are interested in the psychological version of Church's thesis. Our aim is to determinate how the cognitive capacities of humans influences the notion of the intended model.

5   The Intended Model for Arithmetic

The acceptance of Church's thesis results in our first postulate.

Postulate 1 *The intended model for arithmetic have to be recursive.*

Now, the justification of the postulate is obvious. If the model for arithmetic is intended than we should be able to perform the usual arithmetical operations on the elements of this model. But, by Church's thesis, this is only possible when the model is just recursive.

Our second postulate is about the induction axioms.

Postulate 2 *The intended model for arithmetic have to satisfy induction (at least first order induction).*

Of course, if we identify the arithmetic with Peano axioms then this postulate seems to be superfluous. It seems that simply by definition there cannot be a model for arithmetic which do not satisfy induction. Nevertheless, in our approach it is not Peano arithmetic which is considered as a basis for our arithmetical intuitions. What is basic for us are the natural numbers

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[6] Such an approach may be found for example in Shoenfield's book [Sho71].

[7] Two models are recursively isomorphic if there is a recursive function establishing isomorphism between them.

and our computational experience with them. On this ground the principle of induction should be supported by some argument. And indeed, the principle of induction surely was a discovery in the ancient mathematics. Thus, induction requires some form of justification. Of course, such a justification exists.[8] Since this justification is quite known and accepted, we do not go into details of the argument here. We simply mention the fact that the algorithmic definition of the natural numbers is a basis for a justification of induction. Indeed, if natural numbers are exactly the following objects:

– 0 is a natural number,
– if a is a natural number then $a + 1$ is also,

then to show that a given property P is true about all natural numbers it suffices to show the premises of the induction axiom for P. That is $P(0)$ and that if $P(a)$, for a natural number a, then also $P(a + 1)$.

Our second postulate is hence that any model for arithmetic has to satisfy first order induction. Induction together with the Tennenbaum's theorem allows us to describe the class of intended models for arithmetic.

Theorem 1 (Tennenbaum, see [BBJ02]). *Let M be a model of Peano arithmetic. If the interpretation of addition and multiplication in M are recursive then M is a standard model for arithmetic (a model with ω–type ordering).*

Now, if we agree that the intended model should be recursive and should satisfy first order induction, then Tennenbaum's theorem tells us, that it has to have the ω–type ordering. Let us notice, that the ω–type ordering of the intended model is in our paper the conclusion and not a starting point of the reasoning. Moreover, let us observe that if two models satisfy our Postulates 1 and 2 then they are recursively isomorphic. It follows then that the "right" notion of identifications two different models for arithmetic should be recursive isomorphism.

6   Conclusions

Our three assumptions: computability of basic arithmetical operations, the psychological version of Church's thesis and the principle of induction together with Tennenbaum's theorem result in our main conclusion: the intended model for arithmetic is a recursive model with ω–type ordering. Thus, ω–type ordering is now just a corollary of some more fundamental principles, not a basic requirement for being an intended model for arithmetic. The conditions which we postulate define a proper subclass of the standard models for arithmetic that we call intended models. Moreover, it can be easily seen that any two models in this class are recursively isomorphic. Thus, our analysis identifies models for arithmetic only up to recursive isomorphism. We find this quite acceptable. Indeed, if we consider only the border between recursive and nonrecursive then there are hardly any reasons to differentiate between any two recursively isomorphic models.

7   Acknowledgments

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[8] Otherwise, why would we believed in induction?

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